Local Minimizers of the Ginzburg-Landau Functional with Prescribed Degrees

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Abstract

We consider, in a smooth bounded multiply connected domain $\mathcal{D} \subset \mathbb{R}^2$, the Ginzburg-Landau energy $E_\varepsilon(u) = \frac{1}{2}\int_{\mathcal{D}} \left\{|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right\}$ subject to prescribed degree conditions on each component of $\partial\mathcal{D}$. In general, minimal energy maps do not exist [4]. When $\mathcal{D}$ has a single hole, Berlyand and Rybalko [5] proved that for small ε local minimizers do exist. We extend the result in [5]: $E_\varepsilon(u)$ has, in domains $\mathcal{D}$ with $2, 3, ...$ holes and for small ε, local minimizers. Our approach is very similar to the one in [5]; the main difference stems in the construction of test functions with energy control.
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Contents

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1 Introduction

This article deals with the existence problem of local minimizers of the Ginzburg-Landau functional with prescribed degrees in a 2D perforated domain $\mathcal{D}$.

The domain we consider is of the form $\mathcal{D} = \Omega \setminus \cup_{i\in\mathbb{N}_N}\overline{\omega}_i$, where $N \in \mathbb{N}^*$, Ω and the ω_i's are simply connected, bounded and smooth open sets of $\mathbb{R}^2$.

We assume that $\overline{\omega_i} \subset \Omega$ and $\overline{\omega_i} \cap \overline{\omega_j} = \emptyset$ for $i, j \in \mathbb{N}_N := \{1, ..., N\}, i \neq j$.

The Ginzburg-Landau functional is

$$E_\varepsilon(u, \mathcal{D}) := \frac{1}{2}\int_{\mathcal{D}} \left\{|\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2\right\} \mathrm{d}x \tag{1.1}$$

with $u : \mathcal{D} \to \mathbb{C} \simeq \mathbb{R}^2$ and ε is a positive parameter (the inverse of κ, the Ginzburg-Landau parameter).

When there is no ambiguity we will write $E_\varepsilon(u)$ instead of $E_\varepsilon(u, \mathcal{D})$.

Functions we will consider belong to the class

$$\mathcal{J} = \left\{u \in H^1(\mathcal{D}, \mathbb{C}) \,|\, |u| = 1 \text{ on } \partial\mathcal{D}\right\}.$$

Clearly, $\mathcal{J}$ is closed under weak H^1-convergence.

This functional is a simplified version of the Ginzburg-Landau functional which arises in superconductivity (or superfluidity) to model the state of a superconductor submitted to a magnetic field (see, *e.g.*, [10] or [9]). The simplified version of the Ginzburg-Landau functional considered in (1.1) ignores the magnetic field. The issue we consider in this article is existence of local minimizers with prescribed degrees on $\partial\mathcal{D}$.

We next formulate rigorously the problem discussed in this article. To this purpose, we start by defining properly the degrees of a map $u \in \mathcal{J}$. For $\gamma \in \{\partial\Omega, ..., \partial\omega_N\}$ and $u \in \mathcal{J}$ we let

$$\deg_\gamma(u) = \frac{1}{2\pi}\int_\gamma u \times \partial_\tau u \,\mathrm{d}\tau.$$

Here:

- each γ is directly (counterclockwise) oriented,
- $\tau = \nu^\perp$, τ is the tangential vector of γ and ν the outward normal to Ω if $\gamma = \partial\Omega$ or ω_i if $\gamma = \partial\omega_i$,
- $\partial_\tau = \tau \cdot \nabla$, the tangential derivative and " $\cdot$ " stands for the scalar product in $\mathbb{R}^2$,

- $"\times"$ stands for the vectorial product in $\mathbb{C}$, $(z_1+\imath z_2)\times(w_1+\imath w_2) := z_1w_2-z_2w_1$, $z_1, z_2, w_1, w_2 \in \mathbb{R}$,
- the integral over γ should be understood using the duality between $H^{1/2}(\gamma)$ and $H^{-1/2}(\gamma)$ (see, *e.g.*, [4] definition 1).

It is known that $\deg_\gamma(u)$ is an integer see [4] (the introduction) or [7].
We denote the (total) degree of $u \in \mathcal{J}$ in $\mathcal{D}$ by

$$\deg(u, \mathcal{D}) = \left(\deg_{\partial\omega_1}(u), ..., \deg_{\partial\omega_N}(u), \deg_{\partial\Omega}(u)\right) \in \mathbb{Z}^N \times \mathbb{Z}.$$

For $(\mathbf{p}, q) \in \mathbb{Z}^N \times \mathbb{Z}$, we are interested in the minimization of E_ε in

$$\mathcal{J}_{\mathbf{p},q} := \{u \in \mathcal{J} \,|\, \deg(u, \mathcal{D}) = (\mathbf{p}, q)\}\,.$$

There is an huge literature devoted to the minimization of E_ε. In a simply connected domain Ω, the minimization problem of E_ε with the Dirichlet boundary condition $g \in C^\infty(\partial\Omega, \mathbb{S}^1)$ is studied in detail in [6]. E_ε has a minimizer for each $\varepsilon > 0$. This minimizer need not to be unique. In this framework, when $\deg_{\partial\Omega}(g) \neq 0$, the authors studied the asymptotic behaviour of a sequence of minimizers (when $\varepsilon_n \downarrow 0$) and point out the existence (up to subsequence) of a finite set of singularities of the limit.

Other types of boundary conditions were studied, like Dirichlet condition $g \in C^\infty(\partial\Omega, \mathbb{C}\setminus\{0\})$ (in a simply connected domain Ω) in [1] and later for $g \in C^\infty(\partial\Omega, \mathbb{C})$ (see [2]).

If the boundary data is not $u_{|\partial\mathcal{D}}$, but a given set of degrees, then the existence of local minimizers is non trivial. Indeed, one can show that $\mathcal{J}_{\mathbf{p},q}$ is not closed under weak H^1-convergence (see next section), so that one cannot apply the direct method in the calculus of variations in order to derive existence of minimizers. Actually this is not just a technical difficulty, since in general the *infimum* of E_ε in $\mathcal{J}_{\mathbf{p},q}$ is **not** attained, we need more assumptions like the value of the H^1-capacity of $\mathcal{D}$ (see [3] and [4]).

Minimizers u of E_ε in $\mathcal{J}_{\mathbf{p},q}$, if they do exist, satisfy the equation

$$\begin{cases} -\Delta u = \dfrac{u}{\varepsilon^2}(1-|u|^2) & \text{in } \mathcal{D} \\ |u| = 1 & \text{on } \partial\mathcal{D} \\ u \times \partial_\nu u = 0 & \text{on } \partial\mathcal{D} \\ \deg(u, \mathcal{D}) = (\mathbf{p}, q) \end{cases} \tag{1.2}$$

where ∂_ν denotes the normal derivative, *i.e.*, $\partial_\nu = \dfrac{\partial}{\partial\nu} = \nu\cdot\nabla$.

Existence of local minimizers of E_ε is obtained following the same lines as in [5]. It turns out that, even if the *infimum* of E_ε in $\mathcal{J}_{\mathbf{p},q}$ is not attained, (1.2) may have solutions. This was established by Berlyand and Rybalko when $\mathcal{D}$ has a single hole, *i.e.*, when $N = 1$. Our main result is the following generalisation of the main result in [5]:

Theorem 1. *Let $(\mathbf{p}, q) \in \mathbb{Z}^N \times \mathbb{Z}$ and let $M \in \mathbb{N}^*$, there is $\varepsilon_1(\mathbf{p}, q, M) > 0$ s.t. for $\varepsilon < \varepsilon_1$, there are at least M locally minimizing solutions.*

Actually, we will prove a more precise form of Theorem 1 (see Theorem 2), whose statement relies on the notion of *approximate bulk degree* introduced in [5] and generalised in the next section.

The main difference with respect to [5] stems in the construction of the test functions with energy control in section 6. In a sense that will be explained in details in section 6, our construction is local, while the one in [5] is global. We also simplify and unify some proofs in [5].

We do not know whether the conclusion of theorem 1 still holds when $\mathcal{D}$ has no holes at all. That is, we do not know whether for a simply connected domain Ω, a given $d \in \mathbb{Z}^*$ and small ε, the problem

$$\begin{cases} -\Delta u = \dfrac{u}{\varepsilon^2}(1-|u|^2) & \text{in } \Omega \\ u \times \partial_\nu u = 0 & \text{on } \partial\Omega \\ |u| = 1 & \text{on } \partial\Omega \\ \deg_{\partial\Omega}(u) = d & \end{cases} \tag{1.3}$$

has solutions. Existence of a solution of (1.3) is clear when Ω is a disc, say $\Omega = D(0,R)$ (it suffices to consider a solution of $-\Delta u = \frac{u}{\varepsilon^2}(1-|u|^2)$ of the form $u(z) = f(|z|)\left(\dfrac{z}{|z|}\right)^d$ with $u_{|\partial\Omega} = \left(\dfrac{z}{|z|}\right)^d$). However, we do not know the answer when Ω is not radially symmetric anymore.

2 The approximate bulk degree

This section is a straightforward adaptation of [5].

Existence of (local) minimizers for E_ε in $\mathcal{J}_{\mathbf{p},q}$ is not straightforward since $\mathcal{J}_{\mathbf{p},q}$ is not closed under weak H^1-convergence. A typical example (see [4]) is a sequence $(M_n)_n$ s.t.

$$\begin{array}{rccc} M_n: & D(0,1) & \to & D(0,1) \\ & x & \mapsto & \dfrac{x-(1-1/n)}{(1-1/n)x-1} \end{array},$$

where $D(0,1) \subset \mathbb{C}$ is the open unit disc centered at the origin. Then $M_n \rightharpoonup 1$ in H^1, $\deg_{\mathbb{S}^1}(M_n) = 1$ and $\deg_{\mathbb{S}^1}(1) = 0$.

To obtain local minimizers, Berlyand and Rybalko (in [5]) devised a tool: the *approximate bulk degree.* We adapt this tool for a multiply connected domain.

We consider, for $i \in \mathbb{N}_N := \{1, ..., N\}$, V_i the unique solution of

$$\begin{cases} -\Delta V_i = 0 & \text{in } \mathcal{D} \\ V_i = 1 & \text{on } \partial\mathcal{D} \setminus \partial\omega_i \\ V_i = 0 & \text{on } \partial\omega_i \end{cases}. \tag{2.1}$$

For $u \in \mathcal{J}$, we set, noting $\partial_k u = \dfrac{\partial}{\partial x_k} u$

$$\operatorname{abdeg}_i(u, \mathcal{D}) = \frac{1}{2\pi}\int_{\mathcal{D}} u \times (\partial_1 V_i\, \partial_2 u - \partial_2 V_i\, \partial_1 u)\,\mathrm{d}x, \tag{2.2}$$

$$\operatorname{abdeg}(u, \mathcal{D}) = (\operatorname{abdeg}_1(u, \mathcal{D}), ..., \operatorname{abdeg}_N(u, \mathcal{D})).$$

Following [5], we call $\operatorname{abdeg}(u, \mathcal{D})$ the *approximate bulk degree* of u. $\operatorname{abdeg}_i : \mathcal{J} \to \mathbb{R}$, in general, is not an integer (unlike the degree). However, we have

Proposition 1. *1) If $u \in H^1(\mathcal{D}, \mathbb{S}^1)$, then $\operatorname{abdeg}_i(u, \mathcal{D}) = \deg_{\partial\omega_i}(u)$;*

2) Let $\Lambda, \varepsilon > 0$ and $u, v \in \mathcal{J}$ s.t. $E_\varepsilon(u), E_\varepsilon(v) \le \Lambda$, then

$$|\operatorname{abdeg}_i(u) - \operatorname{abdeg}_i(v)| \le \frac{2}{\pi}\|V_i\|_{C^1(\mathcal{D})}\Lambda^{1/2}\|u-v\|_{L^2(\mathcal{D})}; \tag{2.3}$$

3) Let $\Lambda > 0$ and $(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{J}$ s.t. for all $\varepsilon > 0$, $E_\varepsilon(u_\varepsilon) \le \Lambda$, then

$$\operatorname{dist}(\operatorname{abdeg}(u_\varepsilon), \mathbb{Z}^N) \to 0 \text{ when } \varepsilon \to 0. \tag{2.4}$$

Proof of Proposition 1 is postponed to Appendix B.

We define for $\mathbf{d} = (d_1, ..., d_N) \in \mathbb{Z}^N$, $\mathbf{p} = (p_1, ..., p_N) \in \mathbb{Z}^N$ and $q \in \mathbb{Z}$,

$$\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q} = \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}(\mathcal{D}) := \left\{ u \in \mathcal{J}_{\mathbf{p},q} \,|\, \|\text{abdeg}(u) - \mathbf{d}\|_\infty := \max_{i \in \mathbb{N}_N} |d_i - \text{abdeg}_i(u)| \leq \frac{1}{3} \right\}.$$

The following result states that $\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ in never empty for $(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times \mathbb{Z}^N$.

Proposition 2. *Let $(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times \mathbb{Z}^N$. Then $\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q} \neq \emptyset$.*

Proof. For $i \in \{0, ..., N\}$, we denote $\mathbf{e}_i = (\delta_{i,1}, ..., \delta_{i,N}, \delta_{i,0}) \in \mathbb{Z}^{N+1}$ where

$$\delta_{i,k} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} \quad \text{is the Kronecker symbol.}$$

For $i \in \{0, ..., N\}$, there is $M^i_n \in \mathcal{J}_{(p_i - d_i)\mathbf{e}_i}$ if $i \neq 0$ and $M^0_n \in \mathcal{J}_{(q - \sum d_j)\mathbf{e}_0}$ s.t. $M^i_n \rightharpoonup 1$ in H^1 and $|M^i_n| \leq 1$ (Lemmas 6.1 and 6.2 in [4]). Let

$$E_{\mathbf{d}} := \left\{ u \in H^1(\mathcal{D}, \mathbb{S}^1) \,|\, \deg(u, \mathcal{D}) = (\mathbf{d}, d) \right\}, \mathbf{d} = (d_1, ..., d_N),\ d = \sum_{j=1}^N d_j.$$

We note that, $E_{\mathbf{d}} \neq \emptyset$, see, *e.g.*, [6]. Let $u \in E_{\mathbf{d}}$ and $u_n := u \prod_{i=0}^N M^i_n$. Then we will prove that, for large n, we have, up to subsequence, that $u_n \in \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$. Indeed, up to subsequence,

$$u_n \rightharpoonup u \text{ in } H^1,\ \ u_n \in \mathcal{J}_{\mathbf{p},q}.$$

Using the fact that $\text{abdeg}(u) = \mathbf{d}$ and the weak H^1-continuity of the *approximate bulk degree*, we obtain for n sufficiently large, that $u_n \in \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$. $\square$

We denote $m_\varepsilon(\mathbf{p}, q, \mathbf{d})$ the *infimum* of E_ε on $\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$, *i.e*,

$$m_\varepsilon(\mathbf{p}, q, \mathbf{d}) = \inf_{u \in \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}} E_\varepsilon(u)$$

and

$$I_0(\mathbf{d}, \mathcal{D}) = \inf_{u \in E_{\mathbf{d}}} \frac{1}{2} \int_{\mathcal{D}} |\nabla u|^2.$$

We may now state a refined version of Theorem 1.

Theorem 2. *Let $\mathbf{d} \in (\mathbb{N}^*)^N$. Then, for all $(p_1, ..., p_N, q) \in \mathbb{Z}^{N+1}$ s.t. $q \leq d$ and $p_i \leq d_i$, there is $\varepsilon_2 = \varepsilon_2(\mathbf{p}, q, \mathbf{d}) > 0$ s.t. for $0 < \varepsilon < \varepsilon_2$, $m_\varepsilon(\mathbf{p}, q, \mathbf{d})$ is attained.*

Moreover, we have the following estimate

$$m_\varepsilon(\mathbf{p}, q, \mathbf{d}) = I_0(\mathbf{d}, \mathcal{D}) + \pi \left(d_1 - p_1 + ... + d_N - p_N + d - q \right) - o_\varepsilon(1),\ \ o_\varepsilon(1) \underset{\varepsilon \to 0}{\to} 0.$$

For further use, a configuration of degrees $(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times (\mathbb{N}^*)^N$ s.t. $p_i \leq d_i$ and $q \leq \sum d_i$ will be called a "good configuration". Noting that, for $\mathbf{d} \neq \tilde{\mathbf{d}} \in \mathbb{Z}^N$ and $(\mathbf{p}, q) \in \mathbb{Z}^N \times \mathbb{Z}$, we have $\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q} \cap \mathcal{J}^{\tilde{\mathbf{d}}}_{\mathbf{p},q} = \emptyset$, we are led to

Proof of Theorem 1: Let $(\mathbf{p}, q) \in \mathbb{Z}^N \times \mathbb{Z}$ and set for $k \in \mathbb{N}^*$,

$$d = \max \left\{ \max_i |p_i|, |q| \right\} \text{ and } \mathbf{d}_k = (d + k, ..., d + k).$$

We apply Theorem 2 to the class $\mathcal{J}_{\mathbf{p},q}^{\mathbf{d}_k}$. We obtain the existence of

$$\varepsilon_1(\mathbf{p},q,M)=\min_{k\in\mathbb{N}_M}\varepsilon_2(\mathbf{p},q,\mathbf{d}_k)>0$$

s.t. for $\varepsilon<\varepsilon_1$, $k\in\mathbb{N}_M$, $m_\varepsilon(\mathbf{p},q,\mathbf{d}_k)$ is achieved by u_ε^k.

Noting the continuity of the degree and of the *approximate bulk degree* for the strong H^1-convergence, there exists $V_\varepsilon^k\subset\mathcal{J}_{\mathbf{p},q}^{\mathbf{d}_k}\subset\mathcal{J}$ an open (for H^1-norm) neighbourhood of u_ε^k. It follows easily that

$$E_\varepsilon(u_\varepsilon^k)=\min_{u\in V_\varepsilon^k}E_\varepsilon(u).$$

Then $u_\varepsilon^k\in\mathcal{J}_{\mathbf{p},q}$ is a local minimizer of E_ε in $\mathcal{J}$ (for H^1-norm) for $0<\varepsilon<\varepsilon_1(\mathbf{p},q,M)$.

3 Basic facts of the Ginzburg-Landau theory

It is well known (*cf* [4], lemma 4.4 page 22) that the local minimizers of E_ε in $\mathcal{J}_{\mathbf{p},q}$ satisfy

$$-\Delta u=\frac{1}{\varepsilon^2}u(1-|u|^2)\text{ in }\mathcal{D},\tag{3.1}$$

$$|u|=1\text{ and }u\times\partial_\nu u=0\text{ on }\partial\mathcal{D}.\tag{3.2}$$

Equation (3.1) and the Dirichlet condition on the modulus in (3.2) are classical. The Neumann condition on the phase in (3.2) is less standard but it is for example stated in [4].

Equation (3.1) combined with the boundary condition on $\partial\mathcal{D}$ implies, *via* a maximum principle, that

$$|u|\le 1\text{ in }\mathcal{D}.\tag{3.3}$$

One of the questions in the Ginzburg-Landau model is the location of the vortices of stable solutions (*i.e.*, local minimizers of E_ε). We will define *ad hoc* a vortex as an isolated zero x of u with nonzero degree on small circles around x.

The following result shows that, under energy bound assumptions on solutions of (3.1), vortices are expelled to the boundary when $\varepsilon\to 0$.

Lemma 1. *[8] Let $\Lambda>0$ and let u be a solution of (3.1) satisfying (3.3) and the energy bound $E_\varepsilon(u)\le\Lambda$. Then with C, C_k and ε_3 depending only on Λ, $\mathcal{D}$, we have, for $0<\varepsilon<\varepsilon_3$ and $x\in\mathcal{D}$,*

$$1-|u(x)|^2\le\frac{C\varepsilon^2}{\operatorname{dist}^2(x,\partial\mathcal{D})}\tag{3.4}$$

and

$$|D^k u(x)|\le\frac{C_k}{\operatorname{dist}^k(x,\partial\mathcal{D})}.\tag{3.5}$$

When u is smooth in $\mathcal{D}$ and $\rho=|u|>0$, the map $\dfrac{u}{\rho}$ admits a lifting θ , *i.e*, we may write

$$u=\rho e^{\imath\theta},$$

where θ is a smooth (and locally defined) real function on $\mathcal{D}$ and $\nabla\theta$ is a globally defined smooth vector field.

Using (3.1) and (3.2), we have

$$\left\{\begin{array}{ll}\operatorname{div}(\rho^2\nabla\theta)=0 & \text{in } B\\ \partial_\nu\theta=0 & \text{on }\partial\mathcal{D}\end{array}\right.,\tag{3.6}$$

$$\begin{cases} -\Delta\rho + |\nabla\theta|^2\rho + \dfrac{1}{\varepsilon^2}\rho(\rho^2-1) = 0 & \text{in } B \\ \rho = 1 & \text{on } \partial\mathcal{D} \end{cases}, \tag{3.7}$$

here, $B = \{x \in \mathcal{D} \,|\, u(x) \neq 0\}$.

We will need later the following.

Lemma 2. *[5] Let u be a solution of (3.1) and (3.2). Let $G \subset \mathcal{D}$ be an open Lipschitz set s.t. u does not vanish in $\overline{G}$. Write, in $\overline{G}$, $u = \rho v$ with $\rho = |u|$. Let $w \in H^1(G, \mathbb{C})$ be s.t. $|\mathrm{tr}_{\partial G} w| \equiv 1$. Then*

$$E_\varepsilon(\rho w, G) = E_\varepsilon(u, G) + L_\varepsilon(w, G),$$

with

$$L_\varepsilon(w, G) = \frac{1}{2}\int_G \rho^2|\nabla w|^2\,dx - \frac{1}{2}\int_G |w|^2\rho^2|\nabla v|^2\,dx + \frac{1}{4\varepsilon^2}\int_G \rho^4(1-|w|^2)^2\,dx.$$

For further use, we note that we may write, locally in $\overline{G}$, $u = \rho e^{\imath\theta}$, so that $v = e^{\imath\theta}$. It turns out that $\nabla\theta$ is smooth and globally defined in $\overline{G}$. In terms of $\nabla\theta$, we may rewrite

$$L_\varepsilon(w, G) = \frac{1}{2}\int_G \rho^2|\nabla w|^2\,\mathrm{d}x - \frac{1}{2}\int_G |w|^2\rho^2|\nabla\theta|^2\,\mathrm{d}x + \frac{1}{4\varepsilon^2}\int_G \rho^4(1-|w|^2)^2\,\mathrm{d}x.$$

For u a solution of (3.1) and (3.2), we can consider (see Lemma 7 in [5]) h the unique globally defined solution of

$$\begin{cases} \nabla^\perp h = u \times \nabla u & \text{in } \mathcal{D} \\ h = 1 & \text{on } \partial\Omega \\ h = k_i & \text{on } \partial\omega_i \end{cases}, \tag{3.8}$$

where k_i's are real constants uniquely defined by the first two equations in (3.8). Here

$$\nabla^\perp h = \begin{pmatrix} -\partial_2 h \\ \partial_1 h \end{pmatrix} \text{ is the orthogonal gradient of } h \text{ and } u \times \nabla u = \begin{pmatrix} u \times \partial_1 u \\ u \times \partial_2 u \end{pmatrix}.$$

It is easy to show that

$$\begin{cases} \nabla h = -\rho^2\nabla^\perp\theta & \text{in } B \\ \mathrm{div}(\dfrac{1}{\rho^2}\nabla h) = 0 & \text{in } B \\ \Delta h = 2\partial_1 u \times \partial_2 u & \text{in } B \end{cases}; \tag{3.9}$$

here, $B = \{x \in \mathcal{D} \,|\, u(x) \neq 0\}$.

In [6], Bethuel, Brezis and Hélein consider the minimization of $E(u) = \dfrac{1}{2}\displaystyle\int_{\mathcal{D}} |\nabla u|^2\,\mathrm{d}x$, the Dirichlet functional, in the class

$$E_{\mathbf{d}} = \{u \in H^1(\mathcal{D}, \mathbb{S}^1) \,|\, \deg(u, \mathcal{D}) = (\mathbf{d}, d)\};$$

here, $d = \sum d_k$.

Theorem I.1 in [6] gives the existence of a unique solution (up to multiplication by an $\mathbb{S}^1$-constant) for the minimization of E in $E_{\mathbf{d}}$. We denote u_0 this solution. This u_0 is also a solution of

$$\begin{cases} -\Delta v = v|\nabla v|^2 & \text{in } \mathcal{D} \\ v \times \partial_\nu v = 0 & \text{on } \partial\mathcal{D} \end{cases}.$$

Moreover, we have

$$I_0(\mathbf{d}, \mathcal{D}) := \min_{u \in E_{\mathbf{d}}} E(u) = \frac{1}{2}\int_{\mathcal{D}} |\nabla h_0|^2\,\mathrm{d}x \tag{3.10}$$

with h_0 the unique solution of

$$\left\{\begin{array}{cl} \Delta h_0 = 0 & \text{in } \mathcal{D} \\ h_0 = 1 & \text{on } \partial\Omega \\ h_0 = \mathrm{Cst}_k & \text{on } \partial\omega_k,\ k \in \{1, ..., N\} \\ \displaystyle\int_{\partial\omega_k} \partial_\nu h_0 \,\mathrm{d}\sigma = 2\pi d_k & \text{for } k \in \{1, ..., N\} \end{array}\right. . \tag{3.11}$$

One may prove that h_0 is the (globally defined) harmonic conjugate of a local lifting of u_0.

4 Energy needed to change degrees

We denote

$$\begin{array}{rccc} \text{æ}: & (\mathbb{Z}^N \times \mathbb{Z}) \times (\mathbb{Z}^N \times \mathbb{Z}) & \to & \mathbb{N} \\ & ((\mathbf{d}, d), (\mathbf{p}, q)) & \mapsto & \sum_{i=1}^N |d_i - p_i| + |d - q| \end{array} .$$

The next result quantifies the energy needed to change degrees in the weak limit.

Lemma 3. *([4], Lemma 1) Let $(u_n)_n \subset \mathcal{J}_{\mathbf{p},q}$ be a sequence weakly converging in H^1 to u. Then*

$$\liminf_n E(u_n) \geq E(u) + \pi\text{æ}(\deg(u, \mathcal{D}), (\mathbf{p}, q)) \tag{4.1}$$

and for $\varepsilon > 0$

$$\liminf_n E_\varepsilon(u_n) \geq E_\varepsilon(u) + \pi\text{æ}(\deg(u, \mathcal{D}), (\mathbf{p}, q)). \tag{4.2}$$

The next lemma is proved in [5].

Lemma 4. *Let $\mathbf{d} = (d_1, ..., d_N), \mathbf{p} = (p_1, ..., p_N) \in \mathbb{Z}^N$, $q \in \mathbb{Z}$. There is $o_\varepsilon(1) \underset{\varepsilon\to 0}{\to} 0$ (depending of $(\mathbf{p}, q, \mathbf{d})$) s.t. for $u \in \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ we have*

$$E_\varepsilon(u) \geq I_0(\mathbf{d}, \mathcal{D}) + \pi\text{æ}((\mathbf{d}, d), (\mathbf{p}, q)) - o_\varepsilon(1). \tag{4.3}$$

Here, $d := \sum d_i$.

We present below a simpler proof than the original one in [5].

Proof. Let $(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times \mathbb{Z}^N$. We argue by contradiction and we suppose that there are $\delta > 0$, $\varepsilon_n \downarrow 0$ and $(u_n)_n \subset \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ s.t.

$$E_{\varepsilon_n}(u_n) \leq I_0(\mathbf{d}, \mathcal{D}) + \pi\text{æ}((\mathbf{d}, d), (\mathbf{p}, q)) - \delta. \tag{4.4}$$

Since $(u_n)_n$ is bounded in H^1, there is some u s.t., up to subsequence, $u_n \rightharpoonup u$ in H^1 and $u_n \to u$ in L^4. Using the strong convergence in L^4, (4.4) and Proposition 1, we have $u \in H^1(\mathcal{D}, \mathbb{S}^1) \cap \mathcal{J}^{\mathbf{d}}_{\mathbf{d},d} = E_{\mathbf{d}}$.

To conclude, we use (4.4) combined with Lemma 3

$$\begin{array}{rcl} I_0(\mathbf{d}, \mathcal{D}) + \pi\text{æ}((\mathbf{d}, d), (\mathbf{p}, q)) - \delta & \geq & \liminf_n E_{\varepsilon_n}(u_n) \\ & \geq & \liminf_n E(u_n) \\ & \geq & E(u) + \pi\text{æ}((\mathbf{d}, d), (\mathbf{p}, q)) \\ & \geq & I_0(\mathbf{d}, \mathcal{D}) + \pi\text{æ}((\mathbf{d}, d), (\mathbf{p}, q)) \end{array}$$

which is a contradiction. □

One may easily proved (see Lemma 14 in Appendix C) that for $\eta > 0$, $i \in \{0, ..., N\}$ and $u \in \mathcal{J}_{\deg(u,\mathcal{D})}$, there are $v_\pm \in \mathcal{J}_{\deg(u,\mathcal{D})\pm\mathbf{e}_i}$ s.t.

$$E_\varepsilon(v_\pm) \le E_\varepsilon(u) + \pi + \eta.$$

The key ingredient is a sharper result which holds under two additional hypotheses. In order to unify the notations, we use the notation ω_0 for Ω. We may now state the main ingredient in the proof of Theorem 2.

Lemma 5. *Let $u \in \mathcal{J}_{\mathbf{p},q}$ be a solution of (3.1), (3.2).*

Assume that

$$\operatorname{abdeg}_j(u) \in (d_j - \frac{1}{3}, d_j + \frac{1}{3}),\ \forall\, j \in \mathbb{N}_N. \tag{4.5}$$

Let $i \in \{0, ..., N\}$ and assume that there is some point $x^i \in \partial\omega_i$ s.t. $u \times \partial_\tau u(x^i) > 0$. Recall that τ is the direct tangent vector to $\partial\omega_i$.

Then there is $\tilde{u} \in \mathcal{J}_{(\mathbf{p},q)-\mathbf{e}_i}$ s.t.

$$E_\varepsilon(\tilde{u}) < E_\varepsilon(u) + \pi$$

and

$$\operatorname{abdeg}_j(\tilde{u}) \in (d_j - \frac{1}{3}, d_j + \frac{1}{3}),\ \forall j \in \mathbb{N}_N.$$

The proof of Lemma 5 is postponed to section 6.

We also have an upper bound for $m_\varepsilon(\mathbf{p}, q, \mathbf{d})$.

Lemma 6. *Let $\varepsilon > 0$ and $(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times \mathbb{Z}^N$. Then*

$$m_\varepsilon(\mathbf{p}, q, \mathbf{d}) \le I_0(\mathbf{d}, \mathcal{D}) + \pi æ((\mathbf{d}, d), (\mathbf{p}, q)). \tag{4.6}$$

To prove Lemma 6, we need the following

Lemma 7. *Let $u \in \mathcal{J}$, $\varepsilon > 0$ and $\delta = (\delta_1, ..., \delta_N, \delta_0) \in \mathbb{Z}^{N+1}$. For all $\eta > 0$, there is $u^\delta_\eta \in \mathcal{J}_{\deg(u,\mathcal{D})+\delta}$ s.t.*

$$E_\varepsilon(u^\delta_\eta) \le E_\varepsilon(u) + \pi \sum_{i\in\{0,...,N\}} |\delta_i| + \eta \tag{4.7}$$

and

$$\|u - u^\delta_\eta\|_{L^2(\mathcal{D})} = o_\eta(1),\ o_\eta(1) \underset{\eta\to 0}{\to} 0. \tag{4.8}$$

The proof of Lemma 7 is postponed to Appendix C.

Proof. We prove that for $\eta > 0$ small, we have

$$m_\varepsilon(\mathbf{p}, q, \mathbf{d}) \le I_0(\mathbf{d}, \mathcal{D}) + \pi æ((\mathbf{d}, d), (\mathbf{p}, q)) + \eta.$$

We denote $u_0 \in E_{\mathbf{d}}$ s.t. $E(u_0) = I_0(\mathbf{d}, \mathcal{D})$. Then $\operatorname{abdeg}_i(u_0) = d_i$.

Using Lemma 7 with $\delta = (\mathbf{p}, q) - (\mathbf{d}, d)$, there is u_η s.t.

$$u_\eta \in \mathcal{J}_{(\mathbf{p},q)} \text{ and } E_\varepsilon(u_\eta) \le E_\varepsilon(u_0) + \pi æ((\mathbf{d}, d), (\mathbf{p}, q)) + \eta = I_0(\mathbf{d}, \mathcal{D}) + \pi æ((\mathbf{d}, d), (\mathbf{p}, q)) + \eta.$$

Furthermore, by (4.8), $\|u_0 - u_\eta\|_{L^2(\mathcal{D})} = o_\eta(1)$. For η small, by Proposition 1, we have $u_0 \in \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ which proves the lemma. ∎

5 A family with bounded energy converges

In this section we discuss:

1. the asymptotic behaviour of a sequence of solutions of (3.1), (3.2), $(u_{\varepsilon_n})_n \subset \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ ($\varepsilon_n \downarrow 0$) with bounded energy , *i.e*, $E_{\varepsilon_n}(u_{\varepsilon_n}) \leq \Lambda$,
2. the asymptotic behaviour of a minimizing sequence of E_ε in $\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$,
3. a fundamental lemma.

Proposition 3. *Let $\varepsilon_n \downarrow 0$, $(u_{\varepsilon_n})_n \subset \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ with u_{ε_n} a solution of (3.1), (3.2), s.t. for $\Lambda > 0$, we have*

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \leq \Lambda.$$

Then, denoting h_{ε_n} the unique solution of (3.8) with $u = u_{\varepsilon_n}$, we have

$$h_{\varepsilon_n} \rightharpoonup h_0 \text{ in } H^1(\mathcal{D}), \tag{5.1}$$

where h_0 is the unique solution of (3.11).
Up to subsequence, it holds

$$u_{\varepsilon_n} \rightharpoonup u_0 \text{ in } H^1(\mathcal{D}), \tag{5.2}$$

where $u_0 \in E_{\mathbf{d}}$ is the unique solution of (3.10) up to multiplication by an $\mathbb{S}^1$-constant.

Proof. Using the energy bound on u_{ε_n} and a Poincaré type inequality, we have, up to subsequence,

$$h_{\varepsilon_n} \rightharpoonup h \text{ in } H^1.$$

In order to establish (5.1), it suffices to prove that $h = h_0$.

The set $\mathcal{H} := \{h \in H^1(\mathcal{D}, \mathbb{R})\,;\, \partial_\tau h \equiv 0 \text{ on } \partial\mathcal{D} \text{ and } h_{|\partial\Omega} \equiv 1\}$ is closed convex in $H^1(\mathcal{D}, \mathbb{R})$. Since $(h_{\varepsilon_n})_n \subset \mathcal{H}$, we find that $h \in \mathcal{H}$.

By boundedness of $E_{\varepsilon_n}(u_{\varepsilon_n})$, Lemma 1 implies that u_{ε_n} is bounded in $C^2_{\text{loc}}(\mathcal{D}, \mathbb{R}^2)$. Therefore there is some $u \in C^1_{\text{loc}}(\mathcal{D}, \mathbb{C})$ s.t., up to subsequence, $u_{\varepsilon_n} \to u$ in $C^1_{\text{loc}}(\mathcal{D}, \mathbb{R}^2)$, $L^4(\mathcal{D}, \mathbb{R}^2)$ and weakly in $H^1(\mathcal{D}, \mathbb{R}^2)$.

Using the strong convergence in L^4 and the energy bound on u_{ε_n}, we find that $u \in H^1(\mathcal{D}, \mathbb{S}^1)$. It follows that $\partial_1 u \times \partial_2 u = 0$ in $\mathcal{D}$. On the other hand,

$$\Delta h_{\varepsilon_n} = 2\partial_1 u_{\varepsilon_n} \times \partial_2 u_{\varepsilon_n} \to 0 \text{ in } C^0_{\text{loc}}.$$

Therefore, h is a harmonic function in $\mathcal{D}$.

In order to show that $h = h_0$, it suffices to check that

$$\int_{\partial\omega_i} \partial_\nu h \, \mathrm{d}\sigma = 2\pi d_i.$$

To this end, we note that, since $u_{\varepsilon_n} \times (\partial_1 V_i \partial_2 u_{\varepsilon_n} - \partial_2 V_i \partial_1 u_{\varepsilon_n}) = \nabla V_i \cdot \nabla h_{\varepsilon_n}$, we have from (2.1)

$$2\pi \,\text{abdeg}_i(u_{\varepsilon_n}) = \int_{\mathcal{D}} \nabla V_i \cdot \nabla h_{\varepsilon_n} \, \mathrm{d}x \xrightarrow[n\to\infty]{} \int_{\mathcal{D}} \nabla V_i \cdot \nabla h \, \mathrm{d}x = \int_{\partial\mathcal{D}\setminus\partial\omega_i} \partial_\nu h \, \mathrm{d}\sigma.$$

Noting that, by Proposition 1,

$$\begin{cases} \text{abdeg}_i(u_{\varepsilon_n}) \xrightarrow[n\to\infty]{} \text{abdeg}_i(u) = \deg_{\partial\omega_i}(u) \\ \text{abdeg}_i(u_{\varepsilon_n}) \xrightarrow[n\to\infty]{} d_i \end{cases}$$

and that $0 = \int_{\mathcal{D}} \Delta h \,\mathrm{d}x = \int_{\partial\mathcal{D}} \partial_\nu h \,\mathrm{d}\sigma$, we obtain

$$\int_{\partial\mathcal{D}\setminus\partial\omega_i} \partial_\nu h \,\mathrm{d}\sigma = \int_{\partial\omega_i} \partial_\nu h \,\mathrm{d}\sigma = 2\pi\, d_i = 2\pi \deg_{\partial\omega_i}(u).$$

In the first integral, ν is the outward normal to $\mathcal{D}$, in the second, ν is the outward normal to ω_i.

This proves (5.1).

We next turn to (5.2). Let u_0 be s.t., up to subsequence, $u_{\varepsilon_n} \rightharpoonup u_0$ in $H^1(\mathcal{D})$. Since $|u_{\varepsilon_n}| \leq 1$, we find that

$$u_{\varepsilon_n} \times \nabla u_{\varepsilon_n} \rightharpoonup u_0 \times \nabla u_0 \text{ in } L^2(\mathcal{D}).$$

In view of (3.8) and (5.1), we have $u_0 \times \nabla u_0 = \nabla^\perp h_0$. Therefore,

$$E(u_0) = E(h_0) = I_0(\mathbf{d}, \mathcal{D}).$$

Proposition 1 implies that $u_0 \in E_{\mathbf{d}}$. Then u_0 is the unique, up to multiplication by an $\mathbb{S}^1$-constant, minimizer of E in $E_{\mathbf{d}}$. $\square$

Proposition 4. *Let $(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times \mathbb{Z}^N$. For $\varepsilon > 0$, let $(u_n^\varepsilon)_{n\geq 0} \subset \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ be a minimizing sequence of E_ε in $\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$. Then there is $\varepsilon_4(\mathbf{p}, q, \mathbf{d}) > 0$ s.t. for $0 < \varepsilon < \varepsilon_4$, up to subsequence, $u_n \rightharpoonup u$ in H^1 with u which minimizes E_ε in $\mathcal{J}^{\mathbf{d}}_{\deg(u,\mathcal{D})}$.*

Proof. For $\varepsilon > 0$, let $(u_n^\varepsilon)_n \subset \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ be a minimizing sequence of E_ε in $\mathcal{J}$. Up to subsequence, using Proposition 1,

$$u_n^\varepsilon \rightharpoonup u^\varepsilon \text{ in } H^1 \text{ with } u^\varepsilon \in \mathcal{J}^{\mathbf{d}}_{\deg(u^\varepsilon,\mathcal{D})}.$$

Using Lemmas 3 and 6, we see that $\{\deg(u^\varepsilon, \mathcal{D}), \varepsilon > 0\} \subset \mathbb{Z}^N \times \mathbb{Z}$ is a finite set and that $E_\varepsilon(u^\varepsilon)$ is bounded. Therefore, with Proposition 1, there is $\varepsilon_4 > 0$ s.t. $|\mathrm{abdeg}_i(u^\varepsilon) - d_i| < \frac{1}{3}$ for all $i \in \mathbb{N}_N$ and $0 < \varepsilon < \varepsilon_4$.

We argue by contradiction and we assume that there is $\varepsilon < \varepsilon_4$ s.t.

$$E_\varepsilon(u^\varepsilon) = m_\varepsilon(\deg(u^\varepsilon, \mathcal{D}), \mathbf{d}) + 2\eta,\ \eta > 0.$$

Let $u \in \mathcal{J}^{\mathbf{d}}_{\deg(u^\varepsilon,\mathcal{D})}$ be s.t. $E_\varepsilon(u) \leq m_\varepsilon(\deg(u^\varepsilon, \mathcal{D}), \mathbf{d}) + \eta$.

Using Lemma 7 with $\delta = (\mathbf{p}, q) - \deg(u^\varepsilon, \mathcal{D})$, there is $v \in \mathcal{J}_{\mathbf{p},q}$ s.t.

$$E_\varepsilon(v) < E_\varepsilon(u) + \pi æ((\mathbf{p}, q), \deg(u^\varepsilon, \mathcal{D})) + \eta.$$

Furthermore, by (4.8), $\|u - v\|_{L^2}$ can be taken arbitrary small, so that we may further assume $v \in \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$. To summarise we have

$$\begin{array}{rcl} m_\varepsilon(\mathbf{p}, q, \mathbf{d}) & = & \liminf_n E_\varepsilon(u_n^\varepsilon) \\ & \geq & E_\varepsilon(u^\varepsilon) + \pi æ((\mathbf{p}, q), \deg(u^\varepsilon, \mathcal{D})) \\ & = & m_\varepsilon(\deg(u^\varepsilon, \mathcal{D}), \mathbf{d}) + 2\eta + \pi æ((\mathbf{p}, q), \deg(u^\varepsilon, \mathcal{D})) \\ & \geq & E_\varepsilon(u) + \pi æ((\mathbf{p}, q), \deg(u^\varepsilon, \mathcal{D})) + \eta \\ & > & E_\varepsilon(v) \geq m_\varepsilon(\mathbf{p}, q, \mathbf{d}). \end{array}$$

This contradiction completes the proof. $\square$

The main tool requires the following lemma.

Lemma 8. *Let* $(\mathbf{p},q,\mathbf{d})\in\mathbb{Z}^N\times\mathbb{Z}\times\mathbb{Z}^N$ *and* $\Lambda>0$. *There is* $\varepsilon_5(\mathbf{p},q,\mathbf{d},\Lambda)>0$ *s.t. for* $\varepsilon<\varepsilon_5$ *and* $u\in\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$, *a solution of (3.1) and (3.2) with* $E_\varepsilon(u)\le\Lambda$, *if* $d>0$ *(respectively* $d_i>0$*), then there is* $x^0\in\partial\Omega$ *(respectively* $x^i\in\partial\omega_i$*) s.t.* $u\times\partial_\tau u(x^0)>0$ *(respectively* $u\times\partial_\tau u(x^i)>0$*).*

Here τ *is the direct tangent vector to* $\partial\Omega$ *(resp.* $\partial\omega_i$*).*

Proof. We prove existence of $x^0\in\partial\Omega$ under appropriate assumptions. Existence of x^i is similar.

We argue by contradiction. Assume that there are $\varepsilon_n\downarrow 0$, $(u_n)\subset\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$ solutions of (3.1) and (3.2) with $E_{\varepsilon_n}(u_n)\le\Lambda$ s.t. $u_n\times\partial_\tau u_n\le 0$ on $\partial\Omega$.

Since $q=\dfrac{1}{2\pi}\displaystyle\int_{\partial\Omega}u_n\times\partial_\tau u_n$, we have $q\le 0$.

Up to subsequence, by Proposition 3, we can assume that

$$u_n\to u_0 \text{ a.e. with } u_0 \text{ the unique solution (up to } \mathbb{S}^1\text{) of (3.10).}$$

Let $x_0\in\partial\Omega$ and let $\gamma:\partial\Omega\to[0,\mathcal{H}^1(\partial\Omega)[=: I$ be s.t. γ^{-1} is the direct arc-length parametrization of $\partial\Omega$ with the origin at x_0.

We denote $\theta_n: I\to\mathbb{R}$ the smooth functions s.t.

$$\begin{cases} u_n(x)=\mathrm{e}^{\imath\theta_n[\gamma(x)]}\,\forall\, x\in\partial\Omega \\ 0\le\theta_n(0)<2\pi\end{cases}.$$

Then, for all n, θ_n is nonincreasing and $\theta_n\in[\theta_n(0)+2\pi q,\theta_n(0)]\subset[2\pi q,2\pi]$.

Using Helly's selection theorem, up to subsequence, we can assume that $\theta_n\to\theta$ everywhere on I with θ nonincreasing. Denote Ξ the set of discontinuity points of θ. Since θ is nonincreasing, Ξ is a countable set.

Using the monotonicity of θ, we can consider the following decomposition

$$\theta=\theta^c+\theta^\delta,\ \text{with } \theta^c \text{ and } \theta^\delta \text{ are nonincreasing functions.}$$

θ^c is the continuous part of θ and θ^δ is the jump function. The set of discontinuity points of θ^δ is Ξ.

For $t\notin\Xi$,

$$\theta^\delta(t)=\sum_{0<s<t,\,s\in\Xi}\{\theta(s+)-\theta(s-)\}.$$

We obtain easily that $u_0(x)=\mathrm{e}^{\imath\theta[\gamma(x)]}$ a.e. $x\in\partial\Omega$. Since u_0, θ_n and γ have side limits at each points and $u_0=\mathrm{e}^{\imath\theta\circ\gamma}$ a.e., we find that

$$u_0(x\pm)=\mathrm{e}^{\imath\theta[\gamma(x\pm)]} \text{ for } \textbf{each } x\in\partial\Omega.$$

Using the continuity of u_0, we obtain $\mathrm{e}^{\imath\theta[\gamma(x+)]}=\mathrm{e}^{\imath\theta[\gamma(x-)]}\,\forall\, x\in\partial\Omega$ which implies that

$$\theta[\gamma(x+)]-\theta[\gamma(x-)]\in 2\pi\mathbb{Z}\,\forall\, x\in\partial\Omega.$$

For $t\notin\Xi$,

$$\theta^\delta(t)=\sum_{0<s<t,\,s\in\Xi}\{\theta(s+)-\theta(s-)\}\in 2\pi\mathbb{Z}.$$

Then

$$u_0(x)\mathrm{e}^{-\imath\theta^c[\gamma(x)]}=\mathrm{e}^{\imath\theta^\delta[\gamma(x)]}=1 \text{ a.e. } x\in\partial\Omega.$$

Finally, $u_0(x)=\mathrm{e}^{\imath\theta^c[\gamma(x)]}$ a.e. $x\in\partial\Omega$, which is equivalent (using the continuity of the functions) at $u_0=\mathrm{e}^{\imath\theta^c\circ\gamma}$.

We have a contradiction observing that

$$0<2\pi\deg_{\partial\Omega}(u_0)=2\pi d=\theta^c(\mathcal{H}^1(\partial\Omega))-\theta^c(0)$$

and using the fact that θ^c is nonincreasing. □

6 Proof of Lemma 5

We prove only the part of the lemma concerning $\partial\Omega$. The proof for the other connected components of $\partial\mathcal{D}$ is similar.

For reader's convenience, we state the part of Lemma 5 that we will actually prove

Lemma . *Let $u \in \mathcal{J}_{\mathbf{p},q}$ be a solution of (3.1) and (3.2).*

Assume that

$$\text{abdeg}_j(u) \in (d_j - \frac{1}{3}, d_j + \frac{1}{3}), \ \forall\, j \in \mathbb{N}_N \tag{4.5}$$

and that there is some point $x^0 \in \partial\Omega$ s.t. $u \times \partial_\tau u(x^0) > 0$.

Then there is $\tilde{u} \in \mathcal{J}_{(\mathbf{p},q-1)}$ s.t.

$$E_\varepsilon(\tilde{u}) < E_\varepsilon(u) + \pi,$$

$$\text{abdeg}_j(\tilde{u}) \in (d_j - \frac{1}{3}, d_j + \frac{1}{3}), \forall\, j \in \mathbb{N}_N.$$

6.1 Decomposition of $\mathcal{D}$

By hypothesis, there is some $x^0 \in \partial\Omega$ s.t. $\partial_\nu h(x^0) > 0$. Without loss of generality, we may assume that $u(x^0) = 1$.

Then there is $\Upsilon \subset \overline{\mathcal{D}}$, a compact neighbourhood of x^0, simply connected and with nonempty interior, s.t.:

- $\gamma := \partial\Omega \cap \partial\Upsilon$ is connected with nonempty interior;
- x^0 is an interior point of γ;
- $|\nabla h| > 0$, $\rho > 0$, $h \leq 1$ in Υ;
- $\partial_\nu h > 0$ on γ (ν the outward normal to Ω).

It follows that, in Υ, θ, a lifting of $u/|u|$ is globally defined (we take the determination of θ which vanishes at x^0) .

Using the inverse function theorem, we may assume, by further restricting Υ, that there are some $0 < \eta, \delta < 1$ s.t.

$$\Upsilon = \{x \in \mathcal{D} \text{ s.t. } \text{dist}(x, x^0) < \eta,\, 1 - \delta \leq h(x) \leq 1,\, -2\delta \leq \theta(x) \leq 2\delta\}.$$

We may further assume that, by replacing δ by smaller value if necessary and denoting $D_\delta := \overset{\circ}{\Upsilon}$ (see Figure 1), we have

(i) $$\begin{array}{cccc} \Theta := (\theta, h)_{|D_\delta} : & D_\delta & \to & (-2\delta, 2\delta) \times (1-\delta, 1) \\ & x & \mapsto & (\theta, h) \end{array} \quad \text{is a } C^1\text{-diffeomorphism,}$$

(ii) $\partial D_\delta \setminus (\{h = 1\} \cup \{h = 1 - \delta\}) = \partial D_\delta \cap (\{\theta = -2\delta\} \cup \{\theta = 2\delta\})$,

(iii) D_δ is a Lipschitz domain.

We consider $\delta_0 > 0$ s.t. for $\delta < \delta_0$, D_δ satisfies previous properties and

$$|D_\delta|^{1/2} < \frac{\pi \left| \|\text{abdeg}(u) - \mathbf{d}\|_\infty - \frac{1}{3} \right|}{6 \max_i \|V_i\|_{C^1(\mathcal{D})} \left(E_\varepsilon(u) + \pi\right)^{1/2}}. \tag{6.1}$$

Using Proposition 1 and (6.1), if $v \in H^1(\mathcal{D}, \mathbb{C})$ satisfies $u = v$ in $\mathcal{D} \setminus D_\delta$, $|v| \le 2$ in $\mathcal{D}$ and $E_\varepsilon(v) < E_\varepsilon(u) + \pi$, then we have $\mathrm{abdeg}_i(v) \in (d_i - 1/3, d_i + 1/3)$.
We let $\delta < \delta_0$ and we denote

$$D'_\delta := \Theta^{-1}\left[(-\delta, \delta) \times (1-\delta, 1)\right],$$
$$D^-_\delta := \Theta^{-1}\left[(-2\delta, -\delta) \times (1-\delta, 1)\right],$$
$$D^+_\delta := \Theta^{-1}\left[(\delta, 2\delta) \times (1-\delta, 1)\right],$$

so that D'_δ, D^-_δ and D^+_δ are Lipschitz domains (see Figure 1).

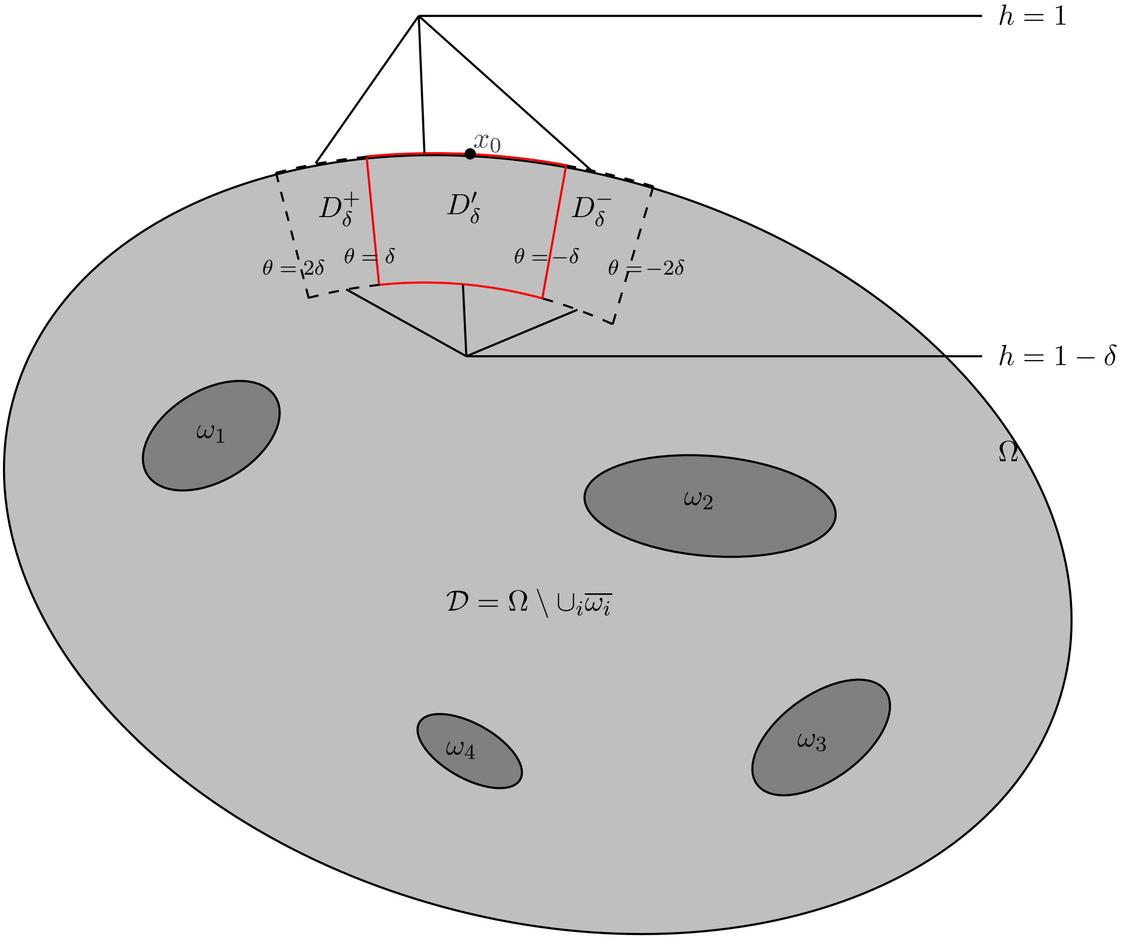

Figure 1: Decomposition of $\mathcal{D}$

6.2 Construction of the test function

We consider an application (with unknown expression in D_δ) $\psi_t : \mathcal{D} \to \mathbb{C}$ ($t > 0$ smaller than δ) s.t.

$$\psi_t(x) = \begin{cases} 1 & \text{in } \mathcal{D} \setminus D_\delta \\ \dfrac{\mathrm{e}^{-\imath\theta} - (1 - t\varphi(\theta))}{\mathrm{e}^{-\imath\theta}(1 - t\varphi(\theta)) - 1} & \text{on } \partial\Omega \cap \partial D_\delta \end{cases}, \tag{6.2}$$

with $0 \le \varphi \le 1$ a smooth, even and 2π-periodic function satisfying

$$\varphi_{|(-\delta/2, \delta/2)} \equiv 1 \text{ and } \varphi_{|[-\pi,\pi[\setminus(-\delta,\delta)} \equiv 0.$$

It is clear that $\psi_{t|\partial\mathcal{D}} \in C^\infty(\partial\mathcal{D})$ and

$$\deg_{\partial\omega_i}(\psi_t) = 0 \text{ for all } i \in \mathbb{N}_N. \tag{6.3}$$

Expanding in Fourier series, we have

$$\frac{\mathrm{e}^{-\imath\theta} - (1 - t\varphi(\theta))}{\mathrm{e}^{-\imath\theta}(1 - t\varphi(\theta)) - 1} = (1 - tb_{-1}(t)) + t \sum_{k \neq -1} b_k(t)\mathrm{e}^{-(k+1)\imath\theta}. \tag{6.4}$$

Noting that the real part of $\dfrac{\mathrm{e}^{-\imath\theta} - (1 - t\varphi(\theta))}{\mathrm{e}^{-\imath\theta}(1 - t\varphi(\theta)) - 1}$ is even and the imaginary part is odd, we obtain that $b_k(t) \in \mathbb{R}$ for all k, t.

The following lemma is proven in Appendix B

Lemma 9. *We denote, for* $\mathrm{e}^{\imath\theta} \in \mathbb{S}^1$,

$$\Psi_t(\mathrm{e}^{\imath\theta}) = \frac{\mathrm{e}^{-\imath\theta} - (1 - t\varphi(\theta))}{\mathrm{e}^{-\imath\theta}(1 - t\varphi(\theta)) - 1} \qquad \textit{and} \qquad \mathcal{F}_t(\mathrm{e}^{\imath\theta}) = \frac{\mathrm{e}^{-\imath\theta} - (1 - t)}{\mathrm{e}^{-\imath\theta}(1 - t) - 1}.$$

Then:

1) $|\Psi_t - \mathcal{F}_t| \leq C_\delta\, t$ *on* $\mathbb{S}^1$*;*

2) $\mathcal{F}_t(z) = \dfrac{\overline{z} - (1 - t)}{\overline{z}(1 - t) - 1} = (1 - tc_{-1}) + t \displaystyle\sum_{k \neq -1} c_k(t)\overline{z}^{k+1}$, *with*

$$c_k = \begin{cases} (t-2)(1-t)^k & \textit{if } k \geq 0 \\ 0 & \textit{if } k \leq -2 \\ 1 & \textit{if } k = -1 \end{cases};$$

3) $|b_k(t) - c_k(t)| \leq C(n, \delta)\,(1 + |k|)^{-n}$, $\forall\, n > 0$ *with* $C(n, \delta)$ *independent of* t *sufficiently small.*

It is easy to see using Lemma 9 that, for t sufficiently small,

$$\deg_{\mathbb{S}^1}(\Psi_t) = \deg_{\mathbb{S}^1}(\mathcal{F}_t) = -1.$$

Using the previous equality and the fact that $\partial_\tau\theta > 0$ on γ, we find that

$$\deg_{\partial\Omega}(\psi_t) = -1. \tag{6.5}$$

It will be convenient to use h and θ as a shorthand for $h(x)$ and $\theta(x)$. With these notations, we will look for ψ_t of the form

$$\begin{aligned} \psi_t(x) &= \tilde{\psi}_t(h, \theta) \\ &= \begin{cases} (1 - tf_{-1}(h)b_{-1}(t)) + t \displaystyle\sum_{k \neq -1} b_k(t)f_k(h)\mathrm{e}^{-(k+1)\imath\theta} & \text{in } D'_\delta \\ \dfrac{\theta - \delta}{\delta} + \tilde{\psi}_t(h, \delta)\dfrac{2\delta - \theta}{\delta} & \text{in } D^+_\delta \\ -\dfrac{\theta + \delta}{\delta} + \tilde{\psi}_t(h, -\delta)\dfrac{2\delta + \theta}{\delta} & \text{in } D^-_\delta \end{cases}. \end{aligned} \tag{6.6}$$

We impose $f_k(1 - \delta) = 0$ and $f_k(1) = 1$ for $k \in \mathbb{Z}$.

Our aim is to show that for $t > 0$ small and appropriate f_k's, the function ψ_t defined by (6.6) satisfies (6.2) and

$$L_\varepsilon(\psi_t\mathrm{e}^{\imath\theta}, D_\delta) < \pi. \tag{6.7}$$

Here, L_ε is the functional defined in Lemma 2, so that

$$E_\varepsilon(\rho\psi_t e^{\imath\theta}, D_\delta) = E_\varepsilon(u, D_\delta) + L_\varepsilon(\psi_t e^{\imath\theta}, D_\delta).$$

Then, considering

$$\underline{\psi_t} = \begin{cases} \psi_t & \text{if } |\psi_t| \le 2 \\ 2\dfrac{\psi_t}{|\psi_t|} & \text{if } |\psi_t| > 2 \end{cases}$$

and setting

$$\tilde{u} = \begin{cases} \rho w_t = \underline{\psi_t} u & \text{in } D_\delta \\ u & \text{in } \mathcal{D} \setminus D_\delta \end{cases},$$

in view of (6.7), it is straightforward that $\tilde{u}$ satisfies the conclusion of Lemma 5.

6.3 Upper bound for $L_\varepsilon(\cdot, D_\delta)$. An auxiliary problem

If we let $\tilde{w} : [1-\delta, 1] \times [-2\delta, 2\delta]$ be s.t. $\tilde{w}(h(x), \theta(x)) := w(x)$, then we have

$$\begin{aligned} |\nabla w|^2 = \sum_i |\partial_i w|^2 &= \sum_i |\partial_h \tilde{w}(h,\theta)\, \partial_i h + \partial_\theta \tilde{w}(h,\theta)\, \partial_i \theta|^2 \\ &= (\rho^4 |\partial_h \tilde{w}(h,\theta)|^2 + |\partial_\theta \tilde{w}(h,\theta)|^2)|\nabla\theta|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} L_\varepsilon(w, D_\delta) &= \frac{1}{2}\int_{D_\delta} \Big\{ \big(\rho^4 |\partial_h \tilde{w}(h,\theta)|^2 + |\partial_\theta \tilde{w}(h,\theta)|^2 - |\tilde{w}(h,\theta)|^2\big)\, \rho^2 |\nabla\theta|^2 + \\ &\qquad + \frac{1}{2\varepsilon^2}\rho^4 (1 - |\tilde{w}(h,\theta)|^2)^2 \Big\}\, \mathrm{d}x \\ &\le \frac{1}{2}\int_{D_\delta} \Big\{ |\partial_h \tilde{w}(h,\theta)|^2 + |\partial_\theta \tilde{w}(h,\theta)|^2 - |\tilde{w}(h,\theta)|^2 + \\ &\qquad + \lambda |e^{\imath\theta} - \tilde{w}(h,\theta)|^2 \Big\} \rho^2 |\nabla\theta|^2\, \mathrm{d}x \qquad (6.8) \\ &=: M_\lambda(w, D_\delta), \end{aligned}$$

provided that $|w| \le 2$ in D_δ and $\lambda \ge \dfrac{9}{2\varepsilon^2 \inf_{D_\delta} |\nabla\theta|^2}$.

In order to simplify formulas, we will write, in what follows, the second integral in (6.8) as

$$\frac{1}{2}\int_{D_\delta} \Big\{ |\partial_h \tilde{w}|^2 + |\partial_\theta \tilde{w}|^2 - |\tilde{w}|^2 + \lambda |e^{\imath\theta} - \tilde{w}|^2 \Big\}\, \rho^2 |\nabla\theta|^2\, \mathrm{d}x.$$

The same simplified notation will be implicitly used for similar integrals.

Remark 1. *If we replace w by $\underline{w} := \frac{w}{|w|}\min(|w|, 2)$, then M_λ does not increase. Furthermore replacing w by $\underline{w}$ does not affect the Dirichlet condition of (6.2). Therefore, by replacing w by $\underline{w}$ if necessary, we may assume $|w| \le 2$.*

We next state a lemma which allows us to give a new form of M_λ.

Lemma 10. *Let $f \in C^1(\mathbb{R}, \mathbb{R})$. Then, for $k \in \mathbb{Z}$, we have*

$$\int_{D'_\delta} f(h)\cos(k\theta)\rho^2 |\nabla\theta|^2\, dx = \begin{cases} 2\delta \displaystyle\int_{1-\delta}^1 f(s)\, ds & \text{if } k = 0 \\ \dfrac{2\sin(k\delta)}{k} \displaystyle\int_{1-\delta}^1 f(s)\, ds & \text{if } k \ne 0 \end{cases},$$

$$\int_{D^\pm_\delta} f(h)\, \rho^2 |\nabla\theta|^2\, dx = \delta \int_{1-\delta}^1 f(s)\, ds.$$

Proof. This result is easily obtained by noting that the jacobian of the change of variable $x \mapsto (\theta(x), h(x))$ is exactly $\rho^2|\nabla\theta|^2$. $\square$

For $w = w_t = \psi_t e^{\imath\theta}$ where ψ_t of the form given by (6.6), we have

$$M_\lambda(w, D_\delta) \;=\; \frac{1}{2}\int_{D_\delta} \left\{|\partial_h \tilde{w}|^2 + |\partial_\theta \tilde{w}|^2 - |\tilde{w}|^2 + \lambda|e^{\imath\theta} - \tilde{w}|^2\right\} \rho^2|\nabla\theta|^2 \, dx.$$

We next rewrite $M_\lambda(w, D'_\delta)$. Recalling that for a sequence $\{a_k\} \subset \mathbb{R}$, we have

$$\left|\sum_{k\in\mathbb{Z}} a_k e^{\imath k\theta}\right|^2 = \sum_{k\in\mathbb{Z}} a_k^2 + 2 \sum_{\substack{k,l\in\mathbb{Z},\\ k>l}} a_k a_l \cos[(k-l)\theta].$$

Then we obtain

$$\begin{aligned}
M_\lambda(w, D'_\delta) \;=\; & \int_{D'_\delta} \Big\{\frac{t^2}{2}\sum_{k\in\mathbb{Z}} {b_k}^2 \left[{f'_k}^2 + {f_k}^2(k^2+\lambda-1)\right] \\
& - t \sum_{k\neq -1} b_k f_k (k+1)\cos[(k+1)\theta] \\
& - t^2 \sum_{k\neq -1} b_{-1} b_k [f'_{-1} f'_k - f_{-1} f_k (k-\lambda+1)] \cos[(k+1)\theta] \\
& + t^2 \sum_{\substack{k,l\neq -1\\ k-l>0}} b_k b_l [f'_k f'_l + (kl+\lambda-1) f_k f_l] \cos[(k-l)\theta] \Big\} \rho^2|\nabla\theta|^2. \qquad (6.9)
\end{aligned}$$

Using Lemma 10 and (6.9), we have

$$\begin{aligned}
M_\lambda(w, D'_\delta) \;=\; & \delta t^2 \sum_{k\in\mathbb{Z}} b_k^2 \phi_k(f_k) - 2t \sum_{k\neq -1} b_k \sin[(k+1)\delta] \int_{1-\delta}^1 f_k \\
& -2t^2 \sum_{k\neq -1} b_{-1} b_k \frac{\sin[(k+1)\delta]}{k+1} \int_{1-\delta}^1 \{f'_{-1} f'_k - (k-\lambda+1) f_{-1} f_k\} \\
& +2t^2 \sum_{\substack{k,l\neq -1\\ k-l>0}} b_k b_l \frac{\sin[(k-l)\delta]}{k-l} \int_{1-\delta}^1 \{f'_k f'_l + (kl+\lambda-1) f_k f_l\} \qquad (6.10) \\
\;=\; & R_\lambda(w) - 2t \sum_{k\neq -1} b_k \sin[(k+1)\delta] \int_{1-\delta}^1 f_k. \qquad (6.11)
\end{aligned}$$

with

$$\begin{aligned}
R_\lambda(w) \;=\; & \delta t^2 \sum_{k\in\mathbb{Z}} b_k^2 \phi_k(f_k) \\
& -2t^2 \sum_{k\neq -1} b_{-1} b_k \frac{\sin[(k+1)\delta]}{k+1} \int_{1-\delta}^1 \{f'_{-1} f'_k - (k-\lambda+1) f_{-1} f_k\} \\
& +2t^2 \sum_{\substack{k,l\neq -1\\ k-l>0}} b_k b_l \frac{\sin[(k-l)\delta]}{k-l} \int_{1-\delta}^1 \{f'_k f'_l + (kl+\lambda-1) f_k f_l\},
\end{aligned}$$

$$\phi_k(f) = \int_{1-\delta}^1 \{f'^2 + \alpha_k^2 f^2\}$$

and

$$\alpha_k = \sqrt{k^2 + \lambda - 1}.$$

We next establish a similar identity for $M_\lambda(w_t, D_\delta^\pm)$. Using (6.6), we have

$$\begin{aligned} M_\lambda(w_t, D_\delta^\pm) &= \frac{1}{2}\int_{D_\delta^\pm} \left\{ |\partial_h \tilde{w}(h,\theta)|^2 + |\partial_\theta \tilde{w}(h,\theta)|^2 - |w|^2 + \lambda |\mathrm{e}^{\imath\theta} - w|^2 \right\} \rho^2 |\nabla\theta|^2 \\ &= \frac{1}{2}\int_{D_\delta^\pm} \left\{ |\partial_h \tilde{\psi}_t(h, \pm\delta)|^2 \left(\frac{2\delta \mp \theta}{\delta}\right)^2 \right. \\ &\quad \left. + \delta^{-2}(1 + \lambda\,(2\delta \mp \theta)^2) |\tilde{\psi}_t(h,\pm\delta) - 1|^2 \mp 2\delta^{-1} \mathrm{Im}\ \tilde{\psi}_t(h,\pm\delta) \right\} \rho^2 |\nabla\theta|^2 \\ &= \frac{1}{2\delta^2}\int_{D_\delta^\pm} \left\{ |\partial_h \tilde{\psi}_t(h,\pm\delta)|^2\,(2\delta \mp \theta)^2 \right. \\ &\qquad \left. + (1 + \lambda\,(2\delta \mp \theta)^2) |\tilde{\psi}_t(h,\pm\delta) - 1|^2 \right\} \rho^2 |\nabla\theta|^2 \\ &\qquad\qquad + t \sum_{k\neq -1} b_k(t) \sin[(k+1)\delta] \int_{1-\delta}^{1} f_k. \end{aligned} \tag{6.12}$$

Here, Im ψ denotes the imaginary part of ψ. To obtain (6.12), we used the identity

$$|\partial_\theta(\psi \mathrm{e}^{\imath\theta})|^2 = |\partial_\theta \psi|^2 + |\psi|^2 + 2\psi \times \partial_\theta \psi.$$

6.4 Choice of $w = \psi_t \mathrm{e}^{\imath\theta}$

We take

$$f_k(h) = \frac{\mathrm{e}^{\alpha_k(h-1)}}{1 - \mathrm{e}^{-2\alpha_k\delta}} + \frac{\mathrm{e}^{-\alpha_k(h-1)}}{1 - \mathrm{e}^{2\alpha_k\delta}}. \tag{6.13}$$

With this choice, by direct computations we have

$$\phi_k(f_k) = \alpha_k \left(1 + \frac{2}{\mathrm{e}^{2\alpha_k\delta} - 1}\right), \tag{6.14}$$

$$\int_{1-\delta}^{1} f_k = \frac{1}{\alpha_k}\left(1 - \frac{2}{\mathrm{e}^{\alpha_k\delta} + 1}\right) \tag{6.15}$$

and for $k, l \in \mathbb{Z}$ s.t. $k \neq \pm l$,

$$\int_{1-\delta}^{1} f_k f_l = \frac{1 - \mathrm{e}^{-2(\alpha_k+\alpha_l)\delta}}{(\alpha_k + \alpha_l)(1 - \mathrm{e}^{-2\alpha_k\delta})(1 - \mathrm{e}^{-2\alpha_l\delta})} - \frac{1 - \mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}}{(\alpha_k - \alpha_l)(1 - \mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta} - 1)}, \tag{6.16}$$

$$\begin{aligned} \frac{1}{\alpha_k\alpha_l}\int_{1-\delta}^{1} f_k' f_l' &= \frac{1 - \mathrm{e}^{-2(\alpha_k+\alpha_l)\delta}}{(\alpha_k + \alpha_l)(1 - \mathrm{e}^{-2\alpha_k\delta})(1 - \mathrm{e}^{-2\alpha_l\delta})} \\ &\quad + \frac{1 - \mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}}{(\alpha_k - \alpha_l)(1 - \mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta} - 1)}. \end{aligned} \tag{6.17}$$

Using (6.11)—(6.17), we may obtain the following estimate, whose proof is postponed to Appendix B.

Lemma 11. *We have*

$$M_\lambda(w_t, D_\delta) \leq \delta - 2\delta t + 4t^2 \sum_{k>l>0} c_k c_l \frac{\sin[(k-l)\delta]}{k-l}\frac{kl}{k+l} + o(t). \tag{6.18}$$

6.5 End of the proof of Lemma 5

We denote

$$S(\delta,t) := \sum_{k>l>0} c_k c_l \frac{\sin[(k-l)\delta]}{k-l}\frac{kl}{k+l}. \tag{6.19}$$

Setting $n = k-l$ and noting that $\dfrac{(n+l)l}{n+2l} = \dfrac{l}{2} + \dfrac{ln}{2(n+2l)}$, we have

$$\frac{2}{(t-2)^2}S(\delta,t) = \sum_{n>0}(1-t)^n\frac{\sin(n\delta)}{n}\sum_{l>0} l(1-t)^{2l} + \sum_{n,l>0}(1-t)^{n+2l}\sin(n\delta)\frac{l}{n+2l}.$$

Here, we have used the explicit formulae for the c_k's, given by Lemma 9.

Using Appendix A (see Appendix A.1) we find that for $0 < t < \delta$, we have

$$\begin{aligned} S(\delta,t) &= \frac{(1-t)^2}{2t^2}\left[\arctan\left(\frac{1-t-\cos\delta}{\sin\delta}\right) + \arctan\left(\frac{\cos\delta}{\sin\delta}\right)\right] \\ &\qquad + \frac{(1-t+\cos\delta)(2-t)}{8t\sin\delta} + \mathcal{O}(1). \end{aligned} \tag{6.20}$$

We note that

$$\begin{aligned} \arctan\left(\frac{1-t-\cos\delta}{\sin\delta}\right) &= \arctan\left(\frac{1-\cos\delta}{\sin\delta}\right) - \frac{t\sin\delta}{2(1-\cos\delta)} + \mathcal{O}(t^2) \\ &= \frac{\delta}{2} - \frac{t\sin\delta}{2(1-\cos\delta)} + \mathcal{O}(t^2) \end{aligned} \tag{6.21}$$

and

$$\arctan\left(\frac{\cos\delta}{\sin\delta}\right) = \frac{\pi}{2} - \delta. \tag{6.22}$$

From (6.20)—(6.22) we infer

$$S(\delta,t) \le \frac{1}{4t^2}(\pi-\delta) + \frac{1}{t}\left[\frac{(1-t+\cos\delta)(2-t)}{8\sin\delta} - \frac{\sin\delta}{4(1-\cos\delta)}\right] + \mathcal{O}(1) \tag{6.23}$$

with

$$\frac{(1-t+\cos\delta)(2-t)}{8\sin\delta} - \frac{\sin\delta}{4(1-\cos\delta)} < \frac{(1+\cos\delta)}{4\sin\delta} - \frac{\sin\delta}{4(1-\cos\delta)} = 0. \tag{6.24}$$

From (6.18),

$$M_\lambda(w_t, D_\delta) \le \delta - 2\delta t + 4t^2 S(\delta,t) + o(t). \tag{6.25}$$

Using (6.23) and (6.24),

$$4t^2 S(\delta,t) \le \pi - \delta + o(t). \tag{6.26}$$

Finally, we have by combining (6.25) with (6.26),

$$M_\lambda(w_t, D_\delta) \le \pi - 2\delta t + o(t) < \pi \text{ for } t \text{ small}. \tag{6.27}$$

We conclude that for t sufficiently small, $L^d_\varepsilon(\underline{w}_t, D_\delta) < \pi$.

6.6 Conclusion

$\tilde{u} := \underline{\psi} u$, with $\underline{\psi} = \psi_t \dfrac{\min(|\psi_t|, 2)}{|\psi_t|}$, satisfies the desired properties *i.e.*:

- $E_\varepsilon(\tilde{u}) < E_\varepsilon(u) + \pi$ (by (6.8) and (6.27)) ;
- $\tilde{u} \in \mathcal{J}^{\mathbf{d}}_{\mathbf{p},q-1}$ (by (6.1), (6.3) and (6.5)).

6.7 A direct consequence of Lemma 5

By applying Lemma 5 and next Lemma 7, one may easily obtain the following

Corollary 1. *Let $u \in \mathcal{J}_{\mathbf{p},q}$ be a solution of (3.1), (3.2).*

Assume that
$$\text{abdeg}_j(u) \in (d_j - \frac{1}{3}, d_j + \frac{1}{3}),\ \forall\, j \in \mathbb{N}_N.$$
Assume that there are $i_0 \in \{0, ..., N\}$ and $x_0 \in \partial\omega_{i_0}$ s.t. $u \times \partial_\tau u(x_0) > 0$.

Then for all $\delta = (\delta_1, ..., \delta_N, \delta_0) \in \mathbb{Z}^{N+1}$ s.t. $\delta_{i_0} > 0$, there is $\tilde{u}_\delta \in \mathcal{J}_{(\mathbf{p},q)-\delta}$ s.t.
$$E_\varepsilon(\tilde{u}_\delta) < E_\varepsilon(u) + \pi \sum_i |\delta_i|$$
and
$$\text{abdeg}_j(\tilde{u}_\delta) \in (d_j - \frac{1}{3}, d_j + \frac{1}{3}),\ \forall j \in \mathbb{N}_N.$$

7 Proof of Theorem 2

The energy estimate is obtained from Lemmas 4 and 6.

We call $(\mathbf{p}, q, \mathbf{d})$ a good configuration of degrees if
$$(\mathbf{p}, q, \mathbf{d}) \in \mathbb{Z}^N \times \mathbb{Z} \times (\mathbb{N}^*)^N,\ p_i \le d_i \text{ and } q \le \sum_i d_i =: d.$$
We first prove Theorem 2 when
$$æ((\mathbf{d}, d), (\mathbf{p}, q)) = |d_1 - p_1| + ... + |d_N - p_N| + |d - q| = 0 \Leftrightarrow \mathbf{p} = \mathbf{d} \text{ and } q = d.$$
For $\varepsilon > 0$, let $(u_n^\varepsilon)_n$ be a minimizing sequence of E_ε in $\mathcal{J}_{\mathbf{d},d}^{\mathbf{d}}$. For $\varepsilon < \varepsilon_4(\mathbf{d}, d, \mathbf{d})$, up to subsequence, using Proposition 4, $u_n^\varepsilon \to u_\varepsilon$ weakly in H^1 and strongly in L^4 and u_ε is a (global) minimizer of E_ε in $\mathcal{J}^{\mathbf{d}}_{\deg(u_\varepsilon,\mathcal{D})}$.

Applying Lemmas 3 and 4, for $\varepsilon < \varepsilon_2(\mathbf{d}) \le \varepsilon_4$ (here, ε_2 is s.t. the $o_\varepsilon(1)$ of Lemma 4 is lower than $\frac{\pi}{2}$),
$$\begin{aligned} I_0(\mathbf{d}, \mathcal{D}) &\ge E_\varepsilon(u_\varepsilon) + \pi æ(\deg(u_\varepsilon, \mathcal{D}), (\mathbf{d}, d)) \\ &\ge I_0(\mathbf{d}, \mathcal{D}) - \frac{\pi}{2} + 2\pi æ(\deg(u_\varepsilon, \mathcal{D}), (\mathbf{d}, d)). \end{aligned}$$
It follows, $æ(\deg(u_\varepsilon, \mathcal{D}), (\mathbf{d}, d)) \le \frac{1}{4}$ which implies $u_\varepsilon \in \mathcal{J}^{\mathbf{d}}_{\mathbf{d},d}$.

We now prove (following the same strategy) Theorem 2 for a good configuration $(\mathbf{p}, q, \mathbf{d})$ s.t.
$$æ((\mathbf{p}, q), (\mathbf{d}, d)) > 0.$$
For $\varepsilon > 0$ consider $(u_n^\varepsilon)_n$ a minimizing sequence of E_ε in $\mathcal{J}^{\mathbf{d}}_{\mathbf{p},q}$.

For $\varepsilon < \varepsilon_4(\mathbf{p}, q, \mathbf{d})$, up to subsequence, using Proposition 4, $u_n^\varepsilon \to u_\varepsilon$ weakly in H^1 and strongly in L^4 and u_ε is a (global) minimizer of E_ε in $\mathcal{J}^{\mathbf{d}}_{\deg(u_\varepsilon,\mathcal{D})}$.

Let $\Lambda := I_0(\mathbf{d}, \mathcal{D}) + æ((\mathbf{p}, q), (\mathbf{d}, d))\pi + 1$, by Lemma 8, for $\varepsilon < \varepsilon_5(\mathbf{p}, q, \mathbf{d}, \Lambda)$, there is $x_\varepsilon^0 \in \partial\Omega$ s.t. $(u_\varepsilon \times \partial_\tau u_\varepsilon)(x_\varepsilon^0) > 0$.

The third assertion in Proposition 1 and the energy bound give the existence of $0 < \varepsilon_2'(\mathbf{p}, q, \mathbf{d}, \Lambda) < \varepsilon_5(\mathbf{p}, q, \mathbf{d}, \Lambda)$ s.t. for $0 < \varepsilon < \varepsilon_2'$,
$$\text{abdeg}_i(u_\varepsilon) \in (d_i - \frac{1}{3}, d_i + \frac{1}{3}).$$

Fix $\varepsilon_2'(\mathbf{p},q,\mathbf{d}) > \varepsilon_2(\mathbf{p},q,\mathbf{d}) > 0$ s.t. the $o_\varepsilon(1)$ in Lemma 4 is lower than $\dfrac{\pi}{2}$ (here ε_5 is defined in Lemma 8).

Using Lemmas 3, 4 and 6, we have for $\varepsilon < \varepsilon_2$

$$\begin{aligned} I_0(\mathbf{d},\mathcal{D}) + \pi æ((\mathbf{p},q),(\mathbf{d},d)) &\geq \liminf E_\varepsilon(u_n^\varepsilon) \text{ (by Lemma 6 and the definition of } (u_n^\varepsilon)_n) \\ &\geq E_\varepsilon(u_\varepsilon) + \pi æ\left((\mathbf{p},q),\deg(u_\varepsilon,\mathcal{D})\right) \quad \text{(Lemma 3)} \\ &\geq I_0(\mathbf{d},\mathcal{D}) + \pi\left[æ\left((\mathbf{p},q),\deg(u^\varepsilon,\mathcal{D})\right)\right. \\ &\qquad \left. + æ\left((\mathbf{d},d),\deg(u^\varepsilon,\mathcal{D})\right)\right] - \frac{\pi}{2} \quad \text{(Lemma 4)} \end{aligned}$$

It follows that

$$æ\left((\mathbf{p},q),\deg(u^\varepsilon,\mathcal{D})\right) + æ\left((\mathbf{d},d),\deg(u^\varepsilon,\mathcal{D})\right) = æ((\mathbf{p},q),(\mathbf{d},d)). \tag{7.1}$$

Thus

$$p_i \leq \deg_{\partial\omega_i}(u^\varepsilon) \leq d_i \text{ and } q \leq \deg_{\partial\Omega}(u^\varepsilon) \leq d.$$

Assume that there is $\varepsilon < \varepsilon_2$ s.t. $u_\varepsilon \notin \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$. Then from Lemma 8 and (7.1), one may apply Corollary 1 to obtain the existence of $\tilde{u}_\varepsilon \in \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$ s.t.

$$m_\varepsilon(\mathbf{p},q,\mathbf{d}) \leq E_\varepsilon(\tilde{u}_\varepsilon) < E_\varepsilon(u_\varepsilon) + \pi æ\left((\mathbf{p},q),\deg(u_\varepsilon,\mathcal{D})\right) \leq \liminf E_\varepsilon(u_n^\varepsilon) = m_\varepsilon(\mathbf{p},q,\mathbf{d})$$

which is a contradiction.

Thus for $\varepsilon < \varepsilon_2$, $u_\varepsilon \in \mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$ and consequently u_ε is a minimizer of E_ε in $\mathcal{J}_{\mathbf{p},q}^{\mathbf{d}}$.

A Results used in the proof of Lemma 5

A.1 Power series expansions

For $X \in \mathbb{C}$, $|X| < 1$, we have

$$\sum_{k\geq 1} \frac{|X|^k}{k} = -\ln(1-|X|), \tag{A.1}$$

$$\sum_{k\geq 0} X^k = \frac{1}{1-X}, \tag{A.2}$$

$$\sum_{k\geq 1} kX^k = \frac{X}{(1-X)^2}, \tag{A.3}$$

$$\sum_{k>0} \sin(k\delta)X^k = \frac{X\sin\delta}{1-2X\cos\delta+X^2}, \tag{A.4}$$

$$\sum_{k>0} \frac{\sin(k\delta)}{k}X^k = \arctan\left(\frac{X-\cos\delta}{\sin\delta}\right) + \arctan\left(\frac{\cos\delta}{\sin\delta}\right), \tag{A.5}$$

$$\sum_{n,l>0} \sin(n\delta)\frac{l}{n+2l}X^{n+2l} = \frac{X+\cos\delta}{4(1-X^2)\sin\delta} - \frac{1}{4\sin^2\delta}\arctan\left(\frac{X-\cos\delta}{\sin\delta}\right) + \mathrm{Cst}(\delta). \tag{A.6}$$

Proof: The first four identities are classical. We sketch the argument that leads to (A.5) and (A.6). The identity (A.5) follows from (A.4) by integration.

We next prove (A.6). Let

$$f(X) = \sum_{n,l>0} \sin(n\delta) \frac{l}{n+2l} X^{n+2l}.$$

On the one hand, by (A.3), (A.4),

$$f'(X) = \frac{1}{X} \sum_{n>0} \sin(n\delta) X^n \sum_{l>0} l X^{2l} = \frac{X^2 \sin\delta}{(1-X^2)^2(1-2X\cos\delta + X^2)}.$$

On the other hand

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{X+\cos\delta}{4\sin\delta(1-X^2)} - \frac{1}{4\sin^2\delta}\arctan\frac{X-\cos\delta}{\sin\delta}\right) = \frac{X^2\sin\delta}{(1-X^2)^2(1-2X\cos\delta+X^2)}.$$

A.2 Estimates for f_k and α_k

Recall that we defined, in section 6, f_k and α_k by

$$f_k(h) = \frac{\mathrm{e}^{\alpha_k(h-1)}}{1-\mathrm{e}^{-2\alpha_k\delta}} + \frac{\mathrm{e}^{-\alpha_k(h-1)}}{1-\mathrm{e}^{2\alpha_k\delta}},$$

$$\alpha_k = \sqrt{k^2+\lambda-1}.$$

In this part, we prove the following inequalities:

$$\alpha_k = |k| + \mathcal{O}\left(\frac{1}{|k|+1}\right), \tag{A.7}$$

$$|f_k(h) - \mathrm{e}^{-|k|(1-h)}| \leq \frac{C}{k^2}, \text{ with } C \text{ independent of } k \in \mathbb{Z}^*,\ h \in (1-\delta, 1), \tag{A.8}$$

$$|f_k'(h) - |k|\mathrm{e}^{-|k|(1-h)}| \leq \frac{C}{|k|}, \text{ with } C \text{ independent of } k \in \mathbb{Z}^*,\ h \in (1-\delta, 1). \tag{A.9}$$

Proof: The first assertion is obtained using a Taylor expansion.

Let $g_h(u) = \mathrm{e}^{u(h-1)}$, we have

$$\begin{aligned} |f_k(h) - \mathrm{e}^{-|k|(1-h)}| &\leq |g_h(\alpha_k) - g_h(|k|)| + \frac{C}{k^2} \\ &\leq \sup_{(|k|,\alpha_k)} |g_h'(u)||\alpha_k - |k|| + \frac{C}{k^2} \leq \frac{1}{\mathrm{e}\,k}\frac{1}{2k} + \frac{C}{k^2} \leq \frac{C}{k^2}. \end{aligned}$$

The proof of (A.9) is similar, one uses $\tilde{g}_h(u) = u\mathrm{e}^{u(h-1)}$ instead of g_h

A.3 Further estimates on f_k and α_k

We have

$$\begin{aligned} 0 \leq \int_{1-\delta}^{1} \left\{ {f_k'}^2 - \alpha_k^2 {f_k}^2 \right\} &\leq \int_{1-\delta}^{1} \left\{ {f_k'}^2 - k^2 {f_k}^2 \right\} \\ &\leq \frac{C}{|k|+1}, \text{ with } C \text{ independent of } k \in \mathbb{Z}, \end{aligned} \tag{A.10}$$

$$\left| \int_{1-\delta}^{1} f_k f_l \right| \leq \frac{C}{\max(|k|,|l|)}, \text{ with } C \text{ independent of } k, l \in \mathbb{Z}, \text{ s.t. } |k| \neq |l|, \tag{A.11}$$

$$\left|\int_{1-\delta}^{1} f_k' f_l'\right| \leq C\left(\min(|k|,|l|)+1\right), \text{ with } C \text{ independent of } k,l \in \mathbb{Z}, \text{ s.t. } |k| \neq |l|. \tag{A.12}$$

Proof: Actually (A.11), (A.12) still hold when $|k| = |l|$, but this will not used in the proof of Lemma 5 and requires a separate argument.

Since $\alpha_k \geq |k|$,

$$\int_{1-\delta}^{1} \left\{ {f_k'}^2 - \alpha_k^2 {f_k}^2 \right\} \leq \int_{1-\delta}^{1} \left\{ {f_k'}^2 - k^2 {f_k}^2 \right\}.$$

By direct computations,

$$0 \leq \int_{1-\delta}^{1} \left\{ {f_k'}^2 - \alpha_k^2 {f_k}^2 \right\} = \frac{4\delta\alpha_k^2}{(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_k\delta}-1)} \leq \frac{C(\delta,n)}{k^n}, \qquad \forall n \in \mathbb{N}^*,$$

$$\int_{1-\delta}^{1} \left\{ {f_k'}^2 - k^2 {f_k}^2 \right\} = \int_{1-\delta}^{1} \left\{ {f_k'}^2 - \alpha_k^2 {f_k}^2 \right\} + (\lambda - 1)\int_{1-\delta}^{1} {f_k}^2,$$

$$\int_{1-\delta}^{1} {f_k}^2 = \frac{1}{2\alpha_k}\left(\frac{1}{1-\mathrm{e}^{-2\alpha_k\delta}} - \frac{1}{1-\mathrm{e}^{2\alpha_k\delta}}\right) + \mathcal{O}\left(\frac{1}{|k|+1}\right) = \mathcal{O}\left(\frac{1}{|k|+1}\right).$$

Which proves (A.10).
For $|k| \neq |l|$, we have

$$\begin{aligned} \left|\int_{1-\delta}^{1} f_k f_l\right| &= \left|\frac{1-\mathrm{e}^{-2(\alpha_k+\alpha_l)\delta}}{(\alpha_k+\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(1-\mathrm{e}^{-2\alpha_l\delta})} - \frac{1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)}\right| \\ &\leq \frac{C}{\max(|k|,|l|)} + \left|\frac{1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)}\right|. \end{aligned}$$

We assume that $|k| > |l|$ and we consider the two following cases: $\alpha_l < \alpha_k \leq 2\alpha_l$ and $\alpha_k > 2\alpha_l$. Noting that $\frac{1-\mathrm{e}^{-2x\delta}}{x}$ is bounded for $x \in \mathbb{R}_+^*$, we have

$$\left|\frac{1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)}\right| \leq \frac{C}{\mathrm{e}^{2\alpha_l\delta}} \leq \frac{C}{\max(|k|,|l|)} \text{ if } \alpha_l < \alpha_k \leq 2\alpha_l,$$

$$\left|\frac{1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)}\right| \leq \frac{C}{\alpha_k-\alpha_l} \leq \frac{C}{\max(|k|,|l|)} \text{ if } \alpha_k > 2\alpha_l.$$

This proves (A.11).
For $|k| \neq |l|$,

$$\int_{1-\delta}^{1} f_k' f_l' = \frac{\alpha_k\alpha_l\left(1-\mathrm{e}^{-2(\alpha_k+\alpha_l)\delta}\right)}{(\alpha_k+\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(1-\mathrm{e}^{-2\alpha_l\delta})} + \frac{\alpha_k\alpha_l\left(1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta}\right)}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)}.$$

It is clear that,

$$\frac{\alpha_k\alpha_l(1-\mathrm{e}^{-2(\alpha_k+\alpha_l)\delta})}{(\alpha_k+\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(1-\mathrm{e}^{-2\alpha_l\delta})} \leq C\frac{\alpha_k\alpha_l}{\alpha_k+\alpha_l} \leq C\left[\min(|k|,|l|)+1\right]. \tag{A.13}$$

As in the proof of (A.11), we have

$$\left|\frac{\alpha_k\alpha_l(1-\mathrm{e}^{-2(\alpha_k-\alpha_l)\delta})}{(\alpha_k-\alpha_l)(1-\mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta}-1)}\right| \leq \frac{C\alpha_k\alpha_l}{\max(|k|,|l|)} \leq C\left[\min(|k|,|l|)+1\right]. \tag{A.14}$$

Inequalities (A.12) follows from (A.13) and (A.14).

A.4 Two fundamental estimates

In this part, we let $k > l \geq 0$ and prove the following:

$$X_{k,l} := \frac{(\alpha_k\alpha_l + kl + \lambda - 1)(1 - \mathrm{e}^{-2(\alpha_k+\alpha_l)\delta})}{(\alpha_k + \alpha_l)(1 - \mathrm{e}^{-2\alpha_k\delta})(1 - \mathrm{e}^{-2\alpha_l\delta})} = \frac{2kl}{k+l} + \mathcal{O}\left(\frac{1}{l+1}\right), \tag{A.15}$$

$$Y_{k,l} := \frac{(\alpha_k\alpha_l + kl + \lambda - 1)(1 - \mathrm{e}^{-2(\alpha_k-\alpha_l)\delta})}{(\alpha_k - \alpha_l)(1 - \mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta} - 1)} \leq C\mathrm{e}^{-\delta l}. \tag{A.16}$$

The computations are direct:

$$\begin{aligned}
X_{k,l} - \frac{2kl}{k+l} &= \frac{2kl}{(\alpha_k + \alpha_l)(1 - \mathrm{e}^{-2\alpha_k\delta})(1 - \mathrm{e}^{-2\alpha_l\delta})} - \frac{2kl}{k+l} + \mathcal{O}\left(\frac{1}{l+1}\right) \\
&= 2kl\frac{k + l - (\alpha_k + \alpha_l)(1 - \mathrm{e}^{-2\alpha_k\delta})(1 - \mathrm{e}^{-2\alpha_l\delta})}{(k+l)(\alpha_k + \alpha_l)(1 - \mathrm{e}^{-2\alpha_k\delta})(1 - \mathrm{e}^{-2\alpha_l\delta})} + \mathcal{O}\left(\frac{1}{l+1}\right) \\
&= \frac{\mathcal{O}\left(k + k^2 l\mathrm{e}^{-l\delta/2}\right)}{(k+l)(\alpha_k + \alpha_l)(1 - \mathrm{e}^{-2\alpha_k\delta})(1 - \mathrm{e}^{-2\alpha_l\delta})} + \mathcal{O}\left(\frac{1}{l+1}\right) \\
&= \mathcal{O}\left(\frac{1}{l+1}\right).
\end{aligned}$$

We now turn to (A.16).

If $\alpha_k \geq 2\alpha_l$ (or equivalently, if $\alpha_k - \alpha_l \geq \frac{\alpha_k}{2}$), then

$$\frac{(\alpha_k\alpha_l + kl + \lambda - 1)(1 - \mathrm{e}^{-2(\alpha_k-\alpha_l)\delta})}{(\alpha_k - \alpha_l)(1 - \mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta} - 1)} \leq C\frac{kl}{\alpha_k}\mathrm{e}^{-2\alpha_l\delta} \leq C\mathrm{e}^{-\delta l}.$$

If $\alpha_k < 2\alpha_l$, then

$$\frac{(\alpha_k\alpha_l + kl + \lambda - 1)(1 - \mathrm{e}^{-2(\alpha_k-\alpha_l)\delta})}{(\alpha_k - \alpha_l)(1 - \mathrm{e}^{-2\alpha_k\delta})(\mathrm{e}^{2\alpha_l\delta} - 1)} \leq Cl^2\mathrm{e}^{-2\alpha_l\delta} \leq C\mathrm{e}^{-\delta l}.$$

B Proof of Proposition 1 and of Lemma 9

B.1 Proof of Proposition 1

The proof of 1) is direct by noting that if $u \in H^1(\mathcal{D}, \mathbb{S}^1)$, then $\partial_1 u$ and $\partial_2 u$ are pointwise proportional and $\deg_{\partial\Omega}(u) = \sum_i \deg_{\partial\omega_i}(u)$,

$$\begin{aligned}
\mathrm{abdeg}_i(u, \mathcal{D}) &= \frac{1}{2\pi}\sum_{k=1,2}(-1)^k\int_{\mathcal{D}}(u \times \partial_k u)\partial_{3-k}V_i \\
&= \frac{1}{2\pi}\int_{\partial\mathcal{D}} V_i\, u \times \partial_\tau u\,\mathrm{d}\tau = \deg_{\partial\Omega}(u) - \sum_{j\neq i}\deg_{\partial\omega_j}(u) = \deg_{\partial\omega_i}(u).
\end{aligned}$$

Proof of 2). Since V_i is locally constant on $\partial\mathcal{D}$, integrating by parts,

$$\int_{\mathcal{D}} v \times (\partial_1 u\,\partial_2 V_i - \partial_2 u\,\partial_1 V_i)\,\mathrm{d}x = \int_{\mathcal{D}} u \times (\partial_1 v\,\partial_2 V_i - \partial_2 v\,\partial_1 V_i)\,\mathrm{d}x.$$

Then

$$\begin{aligned}
2\pi|\mathrm{abdeg}_i(u) - \mathrm{abdeg}_i(v)| &= \Big|\int_{\mathcal{D}}(u - v) \times \Big[(\partial_1 V_i\,\partial_2 u - \partial_2 V_i\,\partial_1 u) \\
&\qquad + (\partial_1 V_i\,\partial_2 v - \partial_2 V_i\,\partial_1 v)\Big]\,\mathrm{d}x\Big| \\
&\leq \sqrt{2}\|u - v\|_{L^2(\mathcal{D})}\|V_i\|_{C^1(\mathcal{D})}(\|\nabla u\|_{L^2(\mathcal{D})} + \|\nabla v\|_{L^2(\mathcal{D})}) \\
&\leq 2\|u - v\|_{L^2(\mathcal{D})}\|V_i\|_{C^1(\mathcal{D})}[E_\varepsilon(u)^{1/2} + E_\varepsilon(v)^{1/2}] \\
&\leq 4\|u - v\|_{L^2(\mathcal{D})}\|V_i\|_{C^1(\mathcal{D})}\Lambda^{1/2}.
\end{aligned}$$

We prove assertion 3) by showing that $\text{dist}(\text{abdeg}_i(u_\varepsilon), \mathbb{Z}) = o(1)$. Using the first and the second assertion, we have

$$\begin{aligned}\text{dist}(\text{abdeg}_i(u_\varepsilon), \mathbb{Z}) &\leq \inf_{v \in E_0^\Lambda} |\text{abdeg}_i(u_\varepsilon) - \text{abdeg}_i(v)| \\ &\leq \frac{2}{\pi} \|V_i\|_{C^1(\mathcal{D})} \Lambda^{1/2} \inf_{v \in E_0^\Lambda} \|u_\varepsilon - v\|_{L^2(\mathcal{D})} \qquad \text{(B.1)}\end{aligned}$$

where $E_0^\Lambda := \left\{ u \in H^1(\mathcal{D}, \mathbb{S}^1) \text{ s.t. } \frac{1}{2} \int_{\mathcal{D}} |\nabla u|^2 \, \mathrm{d}x \leq \Lambda \right\} \neq \emptyset$.

Now, it suffices to show that $\inf_{v \in E_0^\Lambda} \|u_\varepsilon - v\|_{L^2(\mathcal{D})} \to 0$. We argue by contradiction and we assume that there is an extraction $(\varepsilon_n)_n \downarrow 0$ and $\delta > 0$ s.t. for all n, $\inf_{v \in E_0^\Lambda} \|u_{\varepsilon_n} - v\|_{L^2(\mathcal{D})} > \delta$.

We see that $(u_{\varepsilon_n})_n$ is bounded in H^1. Then, up to subsequence, u_n converges to $u \in H^1(\mathcal{D}, \mathbb{R}^2)$ weakly in H^1 and strongly in L^4.

Since $\| |u_{\varepsilon_n}|^2 - 1 \|_{L^2(\mathcal{D})} \to 0$, we have $u \in H^1(\mathcal{D}, \mathbb{S}^1)$ and by weakly convergence, $\|\nabla u\|^2_{L^2(\mathcal{D})} \leq 2\Lambda$.

To conclude, we have $u \in E_0^\Lambda$ et $\|u_{\varepsilon_n} - u\|_{L^2} \to 0$, which is a contradiction.

B.2 Proof of Lemma 9

1) We see easily that, with $z = \mathrm{e}^{\imath\theta}$, we have

$$\frac{\Psi_t(\overline{z}) - \mathcal{F}_t(\overline{z})}{t} = \frac{(1 - \varphi(\theta))(1 - z^2)}{[z(1-t) - 1]\,[z(1 - t\varphi(\theta)) - 1]} \equiv \frac{A(\theta, t)}{B(\theta, t)}. \qquad \text{(B.2)}$$

The modulus of the RHS of (B.2) can be bounded by noting that

- there is some $m > 0$ s.t. $|B(\theta, t)| \geq m$ for each t and each θ s.t. $|\theta| > \delta/2 \bmod 2\pi$;
- there is some $M > 0$ s.t. $|A(\theta, t)| \leq M$ for each t and each θ s.t. $|\theta| > \delta/2 \bmod 2\pi$;
- if $|\theta| \leq \delta/2$ (modulo 2π), then $(\Psi_t - \mathcal{F}_t)\, t^{-1} \equiv 0$.

2) This assertion is a standard expansion.

3) With a classical result relating regularity of $\Psi_t - \mathcal{F}_t$ to the asymptotic behaviour of its Fourier coefficients, we have

$$|b_k(t) - c_k(t)| \leq \frac{2^{n+1}\pi \|\partial_\theta^n (\Psi_t - \mathcal{F}_t)\|_{L^\infty(\mathbb{S}^1)}}{t\,(1 + |k|)^n}.$$

Noting that, for $\partial_\theta^n (\Psi_t - \mathcal{F}_t)\, t^{-1} \equiv \dfrac{A_n(\theta, t)}{B_n(\theta, t)}$

- there is some $m_n > 0$ s.t. $|B_n(\theta, t)| \geq m_n$ for each t and each θ s.t. $|\theta| > \delta/2 \bmod 2\pi$;
- there is some $M_n > 0$ s.t. $|A_n(\theta, t)| \leq M_n$ for each t and each θ s.t. $|\theta| > \delta/2 \bmod 2\pi$;
- if $|\theta| \leq \delta/2$ (modulo 2π), then $(\Psi_t - \mathcal{F}_t)\, t^{-1} \equiv 0$.

Thus the result follows.

B.3 Proof of Lemma 11

The key argument to treat the energetic contribution of $D_\delta^{\pm}$ is the following lemma.

Lemma 12. *1.* $|\tilde{\psi}_t(h,\pm\delta)-1| = \mathcal{O}(t)$*;*

2. $|\partial_h\tilde{\psi}_t(h,\pm\delta)| = \mathcal{O}(t|\ln t|)$.

Proof. (of Lemma 12)
Using Lemma 9, (A.2) and (A.8), we have

$$\begin{aligned} t^{-1}|\tilde{\psi}_t(h,\delta)-1| &\le \left|-c_{-1}f_{-1}(h)+\sum_{k\neq -1} c_k f_k(h)\mathrm{e}^{-\imath[(k+1)\delta]}\right| \\ &\quad + \left|-(b_{-1}-c_{-1})f_{-1}(h)+\sum_{k\neq-1}(b_k-c_k)f_k(h)\mathrm{e}^{-\imath(k+1)\delta}\right| \\ &\le C(\delta)\left\{\left|\sum_{k\ge 0}\left((1-t)\mathrm{e}^{-(1-h)-\imath\delta}\right)^k\right|+1\right\} = \mathcal{O}(1). \end{aligned}$$

We prove that $|\partial_h\tilde{\psi}_t(h,\delta)| = \mathcal{O}(t|\ln t|)$. Using Lemma 9, (A.3) and (A.9),

$$\begin{aligned} t^{-1}|\partial_h\tilde{\psi}_t(h,\delta)| &\le \left|-c_{-1}f'_{-1}+\sum_{k\neq-1}c_k f'_k \mathrm{e}^{-\imath(k+1)\delta}\right| \\ &\quad + \left|-(b_{-1}-c_{-1})f'_{-1}+\sum_{k\neq-1}(b_k-c_k)f'_k\mathrm{e}^{-\imath(k+1)\delta}\right| \\ &\le 2\left|\sum_{k\ge0}k\left[(1-t)\mathrm{e}^{-\imath\delta-(1-h)}\right]^k\right| + \mathcal{O}(|\ln t|) = \mathcal{O}(|\ln t|). \end{aligned}$$

□

Using (6.11), (6.12) and Lemma 12, we have (with the notation of section 6) that

$$M_\lambda(w_t,D_\delta) = R_\lambda(w_t)+o(t),$$

where

$$\begin{aligned} R_\lambda(w_t) &= \delta t^2\sum_{k\in\mathbb{Z}} b_k^2\phi_k(f_k) - 2t^2\sum_{k\neq-1} b_{-1}b_k\frac{\sin[(k+1)\delta]}{k+1}\int_{1-\delta}^1 [f'_{-1}f'_k-(k-\lambda+1)f_{-1}f_k] \\ &\quad + 2t^2\sum_{\substack{k,l\neq-1\\k-l>0}} b_kb_l\frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^1[f'_kf'_l+(kl+\lambda-1)f_kf_l]. \end{aligned}$$

The proof of Lemma 12 is completed provided we establish the following estimate:

$$R_\lambda(w_t) \le \delta - 2\delta t + 4t^2\sum_{\substack{k,l\ge0\\k-l>0}} c_kc_l\frac{\sin[(k-l)\delta]}{k-l}\frac{kl}{k+l}+o(t). \tag{B.3}$$

The remaining part of this appendix is devoted to the proof of (B.3).
We estimate the first term of R_λ:

Using (6.14) and Lemma 9, we have (with C independent of t)

$$\left|\sum_{k\in\mathbb{Z}} b_k^2\phi_k(f_k) - \sum_{k\in\mathbb{Z}} c_k^2\phi_k(f_k)\right| \le C. \tag{B.4}$$

With (6.14) and (A.7), we obtain

$$\phi_k(f_k) \;=\; \alpha(1+\frac{2}{\mathrm{e}^{2\alpha\delta}-1}) = |k| + \mathcal{O}\left(\frac{1}{|k|+1}\right) \text{ when } |k|\to\infty. \tag{B.5}$$

From (A.1), (A.3) and (B.5),

$$\begin{aligned} t^2\sum_{k\in\mathbb{Z}} c_k^2\phi_k(f_k) &= t^2\phi_{-1}(f_{-1}) + t^2(t-2)^2\sum_{k\ge 0}(1-t)^{2k}\phi_k(f_k) \\ &= t^2(t-2)^2\sum_{k>0} k(1-t)^{2k} + o(t) = 1-2t+o(t). \end{aligned} \tag{B.6}$$

We estimate the second term of R_λ:

Using Lemma 9, (A.11) and (A.12), we have (with C independent of t)

$$\left|\sum_{k\neq -1}(b_k-c_k)\frac{\sin[(k+1)\delta]}{k+1}\int_{1-\delta}^1 [f'_{-1}f'_k - (k-\lambda+1)f_{-1}f_k]\right| \le C.$$

Since $b_{-1}(t)$ is bounded by a quantity independent of t, in the order to estimate the third term of the RHS of (6.10), we observe that there is C independent of t s.t.

$$\begin{aligned} \left|\sum_{k\ge 0}(1-t)^k\frac{\sin[(k+1)\delta]}{k+1}\int_{1-\delta}^1 [f'_{-1}f'_k - (k-\lambda+1)f_{-1}f_k]\right| &\le C\left(\sum_{k\ge 1}\frac{(1-t)^k}{k}+1\right) \\ &= C(|\ln t|+1). \end{aligned}$$

Finally, using Lemma 9, (6.16) and (6.17), we have

$$\left|\sum_{k\neq -1} b_k\frac{\sin[(k+1)\delta]}{k+1}\int_{1-\delta}^1 [f'_{-1}f'_k - (k-\lambda+1)f_{-1}f_k]\right| \le C(|\ln t|+1). \tag{B.7}$$

We estimate the last term of R_λ:

First, we consider the case $k=-l>0$ (*i.e.*, $f_k=f_l$). Using (6.15), $0\le f_k\le 1$ and (A.10), we have (with C independent of t)

$$\left|\sum_{k>0} b_k b_{-k}\frac{\sin 2k\delta}{2k}\int_{1-\delta}^1 [f_k'^2 + (-k^2+\lambda-1)f_k^2]\right| \le C.$$

It remains to estimate the last sum in R_λ, considered only over the indices k and l s.t. $|k|\neq|l|$. We start with

$$\begin{aligned} &\sum_{\substack{k,l\neq -1\\ k-l>0,k\neq -l}} (b_kb_l - c_kc_l)\frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^1 [f'_kf'_l + (kl+\lambda-1)f_kf_l] \qquad \text{(B.8)} \\ &= \sum_{\substack{k,l\neq -1\\ k-l>0,k\neq -l}} [(b_k-c_k)(b_l-c_l) + c_k(b_l-c_l) + c_l(b_k-c_k)] * \\ &\qquad * \frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^1 [f'_kf'_l + (kl+\lambda-1)f_kf_l]. \end{aligned}$$

By Assertion 3) of Lemma 9, the first sum of the RHS of (B.8) is easily bounded by a quantity independent of t. By (A.11), (A.12) and Lemma 9,

$$\left|\sum_{\substack{k,l\neq -1\\ k-l>0,k\neq -l}} c_k(b_l-c_l)\frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^1 [f_k'f_l'+(kl+\lambda-1)f_kf_l]\right|$$
$$\leq C\sum_{\substack{k\geq 0,l\neq -1\\ k-l>0,k\neq -l}}\frac{(1-t)^k|b_l-c_l||l|}{k-l}+C.$$

On the other hand (putting $n=k-l$),

$$\begin{aligned}\sum_{\substack{k\geq 0,l\neq -1\\ k-l>0,k\neq -l}}\frac{(1-t)^k|b_l-c_l||l|}{k-l} &\leq \sum_{k>l\geq 0}\frac{(1-t)^k|b_l-c_l|l}{k-l}+\sum_{k\geq 0,l\leq -1}\frac{(1-t)^k|b_l-c_l||l|}{k+|l|}\\ &\leq \sum_{l\geq 0,n>0}\frac{(1-t)^n}{n}|b_l-c_l|l+\sum_{k>0,l\leq -1}\frac{(1-t)^k}{k}|b_l-c_l||l|\\ &= \mathcal{O}(|\ln t|).\end{aligned}$$

Similarly, we may prove that

$$\left|\sum_{\substack{k,l\neq -1\\ k-l>0,k\neq -l}} c_l(b_k-c_k)\frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^1 [f_k'f_l'+(kl+\lambda-1)f_kf_l]\right|=\mathcal{O}(|\ln t|).$$

We have thus proved that

$$\left|\sum_{\substack{k,l\neq -1\\ k-l>0,k\neq -l}} (b_kb_l-c_kc_l)\frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^1 [f_k'f_l'+(kl+\lambda-1)f_kf_l]\right|=o(t^{-1}).$$

To finish the proof, it suffices to obtain

$$\sum_{\substack{k,l\neq -1\\ k-l>0,k\neq -l}} c_kc_l\frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^1 [f_k'f_l'+(kl+\lambda-1)f_kf_l]$$
$$=2\sum_{\substack{k,l\geq 0\\ k-l>0}} c_kc_l\frac{\sin[(k-l)\delta]}{k-l}\frac{kl}{k+l}+o(t^{-1}).$$

Since $c_m=0$ for $m<-1$, it suffices to consider the case $k>l\geq 0$. Under these hypotheses, we have by (6.16), (6.17), (A.15) and (A.16),

$$\begin{aligned}\sum_{k>l\geq 0} c_kc_l\frac{\sin[(k-l)\delta]}{k-l}\int_{1-\delta}^1 [f_k'f_l'+(kl+\lambda-1)f_kf_l] &= 2\sum_{k>l\geq 0} c_kc_l\frac{\sin[(k-l)\delta]}{k-l}\frac{kl}{k+l}\\ &\quad+\mathcal{O}\left(\sum_{k>l\geq 0}\frac{c_kc_l|\sin[(k-l)\delta]|}{k-l}\frac{1}{l+1}\right).\end{aligned}$$

We conclude by noting that

$$\left|\sum_{k>l\geq 0} c_kc_l\left|\frac{\sin[(k-l)\delta]}{(k-l)(l+1)}\right|\right|\leq C\left(1+\sum_{n>0}\frac{(1-t)^n}{n}\sum_{l>0}\frac{(1-t)^{2l}}{l}\right)\leq C(1+\ln^2 t).$$

C Proof of Lemma 7

Lemma 13. *Let $0 < \eta, \delta < 1$, there is*

$$M_{\eta,\delta} : \begin{array}{ccc} D(0,1) & \to & \mathbb{C} \\ x & \mapsto & M_{\eta,\delta}(x) \end{array} \quad s.t.: \tag{C.1}$$

i) $|M_{\eta,\delta}| = 1$ *on* $\mathbb{S}^1$, $\deg_{\mathbb{S}^1}(M_{\eta,\delta}) = 1$,

ii) $\displaystyle\frac{1}{2}\int_{D(0,1)} |\nabla M_{\eta,\delta}|^2 \leq \pi + \eta$,

iii) $|M_{\eta,\delta}| \leq 2$

iv) if $|\theta| > \delta$ *mod* 2π*, then* $M_{\eta,\delta}(\mathrm{e}^{\imath\theta}) = 1$.

Claim: Taking $\overline{M_{\eta,\delta}}$ instead of $M_{\eta,\delta}$, we obtain the same conclusions replacing the assertion *i)* by $\deg_{\mathbb{S}^1}(\overline{M_{\eta,\delta}}) = -1$.

Proof. As in section 6, let $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ be s.t.

- $0 \leq \varphi \leq 1$,
- φ is even and 2π-periodic,
- $\varphi_{|(-\delta/2,\delta/2)} \equiv 1$ and $\varphi_{|[-\pi,\pi[\setminus(-\delta,\delta)} \equiv 0$.

For $0 < t < \delta$, let $M_t = M$ be the unique solution of

$$\left\{\begin{array}{cccl} M(\mathrm{e}^{\imath\theta}) & = & \dfrac{\mathrm{e}^{\imath\theta} - (1 - t\varphi(\theta))}{\mathrm{e}^{\imath\theta}(1 - t\varphi(\theta)) - 1} & \text{on } \partial D(0,1) \\ \Delta M & = & 0 & \text{in } D(0,1) \end{array}\right. .$$

It follows easily that M satisfies *i)*, *iii)* and *iv)*. We will prove that for t small *ii)* holds.

Using (6.4), we have

$$\frac{\mathrm{e}^{\imath\theta} - (1 - t\varphi(\theta))}{\mathrm{e}^{\imath\theta}(1 - t\varphi(\theta)) - 1} = (1 - tb_{-1}(t)) + t\sum_{k\neq -1} b_k(t)\mathrm{e}^{(k+1)\imath\theta}. \tag{C.2}$$

It is not difficult to see that

$$M(r\mathrm{e}^{\imath\theta}) = (1 - tb_{-1}(t)) + t\sum_{k\neq -1} b_k(t) r^{|k+1|}\mathrm{e}^{(k+1)\imath\theta}. \tag{C.3}$$

From (C.3),

$$\begin{aligned} \frac{1}{2}\int_{D(0,1)} |\nabla M|^2 &= t^2 \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r \sum_{k\neq -1} b_k^2 (k+1) r^{2|k+1|-2} \\ &= \pi t^2 \sum_{k\geq 0} b_k^2 (k+1) + \pi t^2 \sum_{k\leq -2} |k+1| b_k^2 \\ &= \pi t^2 \sum_{k\geq 0} c_k^2 (k+1) + \mathcal{O}(t^2) \text{ (using Lemma 9)} \\ &= \pi(2-t)^2 t^2 \sum_{k\geq 0} (1-t)^{2k}(k+1) + \mathcal{O}(t^2) \text{ (using Lemma 9)} \\ &= \pi + \mathcal{O}(t^2) \text{ (using (A.2) and (A.3))} \\ &\leq \pi + \eta \text{ for } t \text{ small.} \end{aligned}$$

We finish the proof taking, for t small, $M_{\eta,\delta} = M_t$. ∎

Lemma 14. *Let $u \in \mathcal{J}$, $i \in \{0, ..., N\}$ and $\varepsilon > 0$. For all $\eta > 0$, there is*

$$u_\eta^\pm \in \mathcal{J}_{\deg(u,\mathcal{D})\pm\mathbf{e}_i}$$

s.t.

$$E_\varepsilon(u_\eta^\pm) \le E_\varepsilon(u) + \pi + \eta \tag{C.4}$$

and

$$\|u - u_\eta^\pm\|_{L^2(\mathcal{D})} = o_\eta(1),\ o_\eta(1) \underset{\eta\to 0}{\to} 0. \tag{C.5}$$

Proof. We prove that for $i = 0$, there is $u_\eta^+ \in \mathcal{J}_{\deg(u,\mathcal{D})+\mathbf{e}_i}$ satisfying (C.4) and (C.5). In the other cases the proof is similar.

Using the density of $C^0(\overline{\mathcal{D}}, \mathbb{C}) \cap \mathcal{J}$ in $\mathcal{J}$ for the H^1-norm, we may assume $u \in C^0(\overline{\mathcal{D}}, \mathbb{C}) \cap \mathcal{J}$.

It suffices to prove the result for $0 < \eta < \min\{10^{-3}, \varepsilon^2\}$.

Let $x^0 \in \partial\Omega$ and V_η be an open regular set of $\mathcal{D}$ s.t. :

- $\partial V_\eta \cap \partial\mathcal{D} \neq \emptyset$, $|V_\eta| \le \eta^2$,
- x^0 is an interior point of $\partial\Omega \cap \partial V_\eta$,
- V_η is simply connected,
- $|u|^2 \le 1 + \eta^2$ in V_η,
- $\|\nabla u\|_{L^2(V_\eta)} \le \eta^2$.

Using the Carathéodory's theorem, there is

$$\Phi : \overline{V_\eta} \to \overline{D(0,1)},$$

a homeomorphism s.t. $\Phi_{|V_\eta} : V_\eta \to D(0,1)$ is a conformal mapping.

Without loss of generality, we may assume that $\Phi(x^0) = 1$. Let $\delta > 0$ be s.t. for $|\theta| \le \delta$ we have $\Phi^{-1}(\mathrm{e}^{\imath\theta}) \in \partial V_\eta \cap \partial\Omega$.

Let $N_\eta \in \mathcal{J}$ be defined by

$$N_\eta(x) = \left\{ \begin{array}{cl} 1 & \text{if } x \in \mathcal{D} \setminus V_\eta \\ M_{\eta^2,\delta}(\Phi(x)) & \text{otherwise} \end{array} \right. .$$

Here, $M_{\eta^2,\delta}$ is defined by Lemma 13. Using the conformal invariance of the Dirichlet functional, we have

$$\frac{1}{2}\int_{V_\eta} |\nabla N_\eta|^2 = \frac{1}{2}\int_{D(0,1)} |\nabla M_{\eta^2,\delta}|^2 \le \pi + \eta^2. \tag{C.6}$$

It is not difficult to see that $u_\eta^+ := uN_\eta \in \mathcal{J}_{\deg(u,\mathcal{D})+\mathbf{e}_0}$. Since $|N_\eta| \le 2$ and $\|N_\eta - 1\|_{L^2(\mathcal{D})} = o_\eta(1)$, using the Dominated convergence theorem, we may prove that $uN_\eta \to u$ in $L^2(\mathcal{D})$ when $\eta \to 0$. It follows that (C.5) holds.

From (C.6) and using the following formula,

$$|\nabla(uv)|^2 = |v|^2|\nabla u|^2 + |u|^2|\nabla v|^2 + 2\sum_{j=1,2}(v\partial_j u)\cdot(u\partial_j v)$$

we obtain

$$\begin{aligned} \frac{1}{2}\int_{V_\eta} |\nabla u_\eta^+|^2 &= \frac{1}{2}\int_{V_\eta} \left\{ |N_\eta|^2|\nabla u|^2 + |u|^2|\nabla N_\eta|^2 + 2\sum_{j=1,2}(N_\eta\partial_j u)\cdot(u\partial_j N_\eta) \right\} \\ &\le (1+\eta^2)(\pi+\eta^2) + 2\|\nabla u\|_{L^2(V_\eta)}^2 + 4\sqrt{1+\eta^2}\|\nabla u\|_{L^2(V_\eta)}\|\nabla N_\eta\|_{L^2(V_\eta)} \\ &\le \pi + \frac{\eta}{2}. \end{aligned} \tag{C.7}$$

Furthermore, we have

$$\frac{1}{4\varepsilon^2}\int_{V_\eta}(1-|u_\eta^+|^2)^2 \le \frac{\eta^2}{4\varepsilon^2} \le \frac{\eta}{2}. \tag{C.8}$$

From (C.7) and (C.8), it follows

$$E_\varepsilon(u_\eta^+,\mathcal{D}) = E_\varepsilon(u,\mathcal{D}\setminus V_\eta) + E_\varepsilon(u_\eta^+,V_\eta) \le E_\varepsilon(u,\mathcal{D}) + \pi + \eta.$$

The previous inequality completes the proof. □

We may now prove Lemma 7. For the convenience of the reader, we recall the statement of the lemma.

Lemma . *Let $u \in \mathcal{J}$, $\varepsilon > 0$ and $\delta = (\delta_1, ..., \delta_N, \delta_0) \in \mathbb{Z}^{N+1}$. For all $\eta > 0$, there is $u_\eta^\delta \in \mathcal{J}_{\deg(u,\mathcal{D})+\delta}$ s.t.*

$$E_\varepsilon(u_\eta^\delta) \le E_\varepsilon(u) + \pi \sum_{i\in\{0,\dots,N\}} |\delta_i| + \eta \tag{4.7}$$

and

$$\|u - u_\eta^\delta\|_{L^2(\mathcal{D})} = o_\eta(1),\ o_\eta(1) \underset{\eta\to 0}{\to} 0. \tag{4.8}$$

Proof. As in the previous lemma, it suffices to prove the proposition for $0 < \eta < \min\{10^{-3}, \varepsilon^2\}$ and $u \in C^0(\overline{\mathcal{D}},\mathbb{C}) \cap \mathcal{J}$.

We construct u_η^δ in $\ell_1 = \sum_{i\in\{0,\dots,N\}} |\delta_i|$ steps. If $\ell_1 = 0$ (which is equivalent at $\delta = \mathbf{0}_{\mathbb{Z}^{N+1}}$) then, taking $u_\eta^\delta = u$, (4.7) and (4.8) hold.

Assume $\ell_1 \neq 0$. Let $\Gamma = \{i \in \{0, ..., N\} \mid \delta_i \neq 0\} \neq \emptyset$, $L = \operatorname{Card}\Gamma$ and $\mu = \dfrac{\eta}{\ell_1}$. We enumerate the elements of Γ in $(i_n)_{n\in\mathbb{N}_L}$ s.t. for $n \in \mathbb{N}_{L-1}$ we have $i_n < i_{n+1}$.

Let σ be the sign function *i.e.* for $x \in \mathbb{R}^*$, $\sigma(x) = \dfrac{x}{|x|}$.

For $n \in \mathbb{N}_L$ and $l \in \mathbb{N}_{|\delta_{i_n}|}$, we construct

$$v_n^l \in \mathcal{J}_{\deg(v_n^{l-1},\mathcal{D})+\sigma(\delta_i)\mathbf{e}_{i_n}}$$

s.t

$$v_0^0 = u,\ v_n^0 = v_{n-1}^{|\delta_{i_{n-1}}|} \text{ with for } n=1,\ \delta_{i_0}=0,$$
$$v_n^{l+1} = \left\{\begin{array}{ll} (v_n^l)_\mu^+ & \text{if } \delta_{i_n} > 0 \\ (v_n^l)_\mu^- & \text{if } \delta_{i_n} < 0 \end{array}\right.,\ 0 \le l < |\delta_{i_n}| .$$

Here, $(v_n^l)_\mu^\pm$ stands for $u_\mu^\pm$ defined by Lemma 14 taking $u = v_n^l$ and $\eta = \mu$.

It is clear that v_n^l is well defined and that for $n \in \mathbb{N}_L$, $v_n := v_n^{|\delta_{i_n}|} \in \mathcal{J}_{\deg(v_{n-1},\mathcal{D})+\delta_{i_n}\mathbf{e}_{i_n}}$ with $v_0 = u$.

Therefore, using (C.4), we have for $n \in \mathbb{N}_L$,

$$v_n \in \mathcal{J}_{\deg(u,\mathcal{D})+\sum_{k\in\mathbb{N}_n}\delta_{i_k}\mathbf{e}_{i_k}},\ E_\varepsilon(v_n) \le E_\varepsilon(u) + (\pi+\mu)\sum_{k\in\mathbb{N}_n}|\delta_{i_k}|.$$

Taking $n = L$, we obtain that

$$u_\eta^\delta = v_L \in \mathcal{J}_{\deg(u,\mathcal{D})+\delta},\ E_\varepsilon(u_\eta^\delta) \le E_\varepsilon(u) + \pi\sum_{i\in\{0,\dots,N\}}|\delta_i| + \eta.$$

Furthermore, u_η^δ is obtained from u multiplying by ℓ_1 factors N_l, $l \in \mathbb{N}_{\ell_1}$. Each N_l is bounded by 2 and converges to 1 in L^2-norm (when $\eta \to 0$). Using the Dominated convergence theorem, we may prove that u_η^δ satisfies (4.8).

□

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